

EXTERIOR  
BALLISTICS

1935

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HERRMANN

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## PREFACE

*Exterior Ballistics, 1935* has been prepared to serve principally as a textbook for the instruction of midshipmen at the United States Naval Academy, and its scope is confined to material which is appropriate for the training of naval officers in the problems of naval gunnery. Although a number of chapters deal with fundamentals that are general in their application to the problems of exterior ballistics, complete development of practical methods is confined to those which are, at the time of this writing, considered standard in the United States Navy. The notation, likewise, is in accordance with usage in the United States Navy. These features limit the usefulness of this volume as a general reference book, and it is not intended that it should be regarded as such.

In the preparation of this volume, close attention has been given to the limitations imposed upon it by its special purpose. These limitations apply both to the time allotted to the subject in the curriculum of the United States Naval Academy and to the mathematical equipment of the students for whom the book is designed. In view of the limited time available to midshipmen for the preparation of lessons, an effort has been made to include in the text thorough explanations of features which, in the past, have proved troublesome and have resulted in a drain on recitation periods. In general, the formulas given in the text are preceded by complete derivations, in order that the student may have full proof of the methods employed without being obliged to consult reference books. In a few cases, however, this would lead to digressions of greater length than justified by the ends to be gained; in such cases reference is made to sources in which details omitted from the text may be found.

Although its principal purpose is to serve as a textbook for undergraduate study, this volume has been designed to serve also as a reference book for those who desire to extend their studies of exterior ballistics beyond the scope of the undergraduate course. Treatments of certain special features are outlined in footnotes for this purpose. Also, reference is made to numerous sources which are appropriate for collateral study. It is not intended, however, that consultation of these sources should constitute a part of the undergraduate course.

The various tables that are required in connection with this textbook are published as a separate volume, entitled *Range and Ballistic Tables, 1935*. The contents of the latter volume are stated in its introductory pages.

*Exterior Ballistics, 1935* is the successor to *Exterior Ballistics, 1930* and *Exterior Ballistics, 1926*, which were prepared by the author of the present volume. Although much of the material appearing in the 1926 and 1930 editions has been retained in the present volume, the latter has been completely rewritten and is essentially a new textbook. The 1926 and 1930 editions covered in detail the computation of range tables by both the Ingalls-Siacci Method and the numerical-integration method. Siacci's Method, with its various outgrowths, was the primary method for the computation of the U.S. Navy range tables from about 1890 until about 1920; since about 1920, Siacci's Method has been restricted to the computation of range-table values for trajectories having angles of departure not exceeding  $15^\circ$ , and the numerical-integration method has been used in connection with trajectories having angles of departure greater than  $15^\circ$ . It is anticipated that the numerical-integration method will, in the near future, replace Siacci's Method altogether. Accordingly, the latter is treated in the present volume as an obso-

lescent method, and the practical features involved in the computation of modern range tables are dealt with primarily according to the numerical-integration method. The scope of the book has been extended to include practical methods for dealing with the effects of variations in the assumed standard air structure and of non-uniform winds, as determined by aloft soundings, and with the effects of the earth's rotation. Advantage has been taken of the extensive revision outlined above to modernize the subject matter throughout and to condense and simplify the presentation.

A complete statement of those to whom acknowledgment for contributions to this textbook is due would involve the naming of individuals who, during a period of almost fifty years, have been the authors of and contributors to the United States Naval Academy textbook on exterior ballistics. The first of these textbooks was *Exterior Ballistics, 1887*, by Lieutenant J. F. Meigs, U.S. Navy, and Lieutenant R. R. Ingersoll, U.S. Navy, and it was succeeded, in turn, by the following: *Exterior Ballistics, 1893* and *Exterior Ballistics, 1901*, by Lieutenant Commander R. R. Ingersoll, U.S. Navy; *Exterior Ballistics, 1904* and *Exterior Ballistics, 1906*, by Professor P. R. Alger, U.S. Navy; *The Groundwork of Practical Naval Gunnery, or Exterior Ballistics, 1915*, by Professor P. R. Alger, U.S. Navy; and *Exterior Ballistics, 1926* and *Exterior Ballistics, 1930*, by the author of the present volume. Although the 1915 book appeared as a revision of Professor Alger's previous works, it represented, in fact, a very considerable extension beyond the scope of the latter, and was actually prepared by Captain L. H. Chandler, U.S. Navy. Captain Chandler's book, in many respects, was used as a model by the present author. Acknowledgments are due, and are gratefully made, to all of the above mentioned sources and to the contributors mentioned by their respective authors, as well as to numerous other sources to which reference is made at appropriate points in the text.

*Exterior Ballistics, 1926* was prepared under the immediate direction of Captain Walter S. Anderson, U.S. Navy, then Head of Department of Ordnance and Gunnery, U.S. Naval Academy. The author owes the accomplishment of his tasks in connection with the 1926, 1930, and 1935 editions of this textbook in large measure to the continued confidence and encouragement of Captain Anderson.

The author is indebted also to the following for immediate assistance in the preparation of the present volume: Captain G. L. Schuyler, U.S. Navy, Bureau of Ordnance, U.S. Navy Department, for helpful criticism of the 1926 and 1930 editions; Mr. J. W. Webb, Bureau of Ordnance, U.S. Navy Department, and Mr. S. Feltman, Technical Staff, Ordnance Department, U.S. Army, for advice on certain technical details. The author has sought to embody in this work the suggestions of all instructors in the Department of Ordnance and Gunnery, U.S. Naval Academy, principally as to the manner of presenting features which have in the past given difficulty to the student, and is, to this extent, indebted to all officers who have served with him in this Department.

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1 September 1935.

## CHAPTER 1

### INTRODUCTION

101. The science of exterior ballistics has for centuries engaged some of the world's best mathematical genius. A treatise on the flight of projectiles was published by Nicholas Tartaglia in 1537, and this may be regarded as the beginning of what has since become a very extensive literature on this subject. Galileo contributed to the investigations of the trajectory in vacuum, and Newton considered the effects of air resistance on the trajectory. As early as 1753, Euler gave an approximate solution of the trajectory in air that is of particular interest because of its close approach to the only recently adopted methods of our own day. These early treatments of the problem, and the almost innumerable ones that have followed them and have lead to the methods now in use, form a classical background well worth study by students so inclined.

The aim of this textbook is to treat the science of exterior ballistics from the viewpoint of its practical applications to the problems of modern naval gunnery. Occasionally it is useful, in this connection, to trace historically the development of some features, in order that the line of progress may be indicated. The time available for the study of this course at the Naval Academy is not sufficient to permit extensive consideration of alternative methods. The methods explained in this text are limited, in general, to those employed in the U.S. Navy. However, reference is made frequently to sources in which unlimited treatments may be found.

102. The ultimate purpose of this textbook is to teach naval officers how to direct the fire from the guns of a ship in the most effective manner. In order to accomplish this purpose, the problem is treated in three stages, as follows:

- A. First to be considered is the problem of solving the trajectory in air. In this stage we find occasion to deal extensively with theory. Two distinct problems are met in this portion of our study, namely, (1) investigation and measurement of the forces which operate on a projectile in flight, and (2) the establishment of methods by means of which all of these forces may be accounted for in a solution of the trajectory, with a degree of accuracy commensurate with the requirements of practice. This stage embraces also the development of tables by means of which the solution of a trajectory may be facilitated; such tables are called *ballistic tables*. This stage is not confined to any particular branch of gunnery, but deals with fundamental applications of the theories of exterior ballistics to the solution of the trajectory.
- B. Next to be considered is the problem of reducing the general information contained in ballistic tables to the specific information required for any given gun for all circumstances attending its use. This problem is handled by preparing for each gun a *range table* in which all data required for the control of that gun are tabulated in convenient form. In this stage we shall limit ourselves to the problems involved in constructing range tables in the form in which the latter are used in the U.S. Navy. The range

tables used by other services may differ materially, in form, from our own.

C. Finally we shall deal with the problem of using range tables for the determination of firing data, and with the application of the laws of errors to the analysis of results. In this final stage we shall deal with the immediate, practical applications of the science of exterior ballistics to the problems involved in the control of gunfire from ships.

In actual practice it is ordinarily not necessary to deal specifically with matters outlined in the first two stages; each ship is provided with range tables for its guns, and the ordinary shipboard problem is one of using these tables. It is to be expected, however, that a more intelligent use of range tables will result from prior consideration of the problems entering into their construction. Also, the possibility of having to use a ship's guns in a manner not provided for by available range tables should not be overlooked. Such instances have occurred in the past. Complete mastery of a ship's armament, under any circumstances that may arise, demands the ability to prepare firing data from the ground up, if necessary.

103. The methods of exterior ballistics may be called semi-empirical. No exact solution of a trajectory in air exists, and it is not probable that one will ever be devised. The factors governing the flight of a projectile are numerous and, even under the best of conditions, some of these factors are of an uncertain nature. We refer to accurate solutions and to approximate solutions, but these terms are only relative. The accuracy of any solution is limited ultimately by the precision with which the physical values entering into it can be measured; some of these will probably always remain indeterminable to a certain degree. In vacuum, the trajectory is a true parabola and its exact solution offers no difficulty. The presence of an air medium introduces many complications; some of these can be handled effectively only by deducing from the results of actual firing empirical constants which may be used in connection with mathematical solutions.

104. Numerical exercises in connection with the first two stages outlined in article 102 (embracing Chapters 2 to 11) are introduced in order to ensure familiarity with the sources of data for making trajectory solutions, as well as to promote that degree of understanding of processes which can come only from actually using them. The problem of assigning appropriate standards of computational accuracy in connection with such exercises has always been a troublesome one. But despite the various considerations that may be advanced in support of greater latitude in the matter, adherence to uniform methods and standards is required in the solutions submitted by midshipmen in order that errors in the solutions may be traced readily. In the strictly practical exercises of the third stage outlined in article 102, the considerations discussed above apply equally well, and, in addition, correctness of the result is of itself an important end, for these exercises are to be regarded as training for similar work to be done in actual service. The real significance of accuracy in a strictly practical problem in gunnery often depends largely upon the immediate circumstances of the case involved. It is well known that in actual practice unaccountable errors often occur. But the fact that a target is often missed on the first salvo despite the best of efforts to determine the ballistic corrections accurately, does not render such efforts futile, any more than the fact that unknown ocean currents often carry a ship from her designed course renders accurate navigational methods futile. Actual experience is the only safe guide as to the degree of accuracy that is appropriate

under a given set of conditions. The midshipman, in working the exercises given in this book, is expected to adhere to the standards of accuracy that are used in the solutions illustrated in the text.

### LETTERS OF THE GREEK ALPHABET USED AS SYMBOLS.

Letter. Pronunciation.	Letter. Pronunciation.	Letter. Pronunciation.
$\alpha$ . . . . Alpha.	$\theta$ . . . . Theta.	$\Sigma$ or $\sigma$ . . . . Sigma.
$\beta$ . . . . Beta.	$\lambda$ . . . . Lambda.	$\phi$ . . . . Phi.
$\gamma$ . . . . Gamma.	$\mu$ . . . . Mu.	$\psi$ . . . . Psi.
$\Delta$ or $\delta$ . . . . Delta.	$\pi$ . . . . Pi.	$\Omega$ or $\omega$ . . . . Omega.
$\epsilon$ . . . . Epsilon.	$\rho$ . . . . Rho.	

## CHAPTER 2

### DEFINITIONS AND INTRODUCTORY EXPLANATIONS. PRELIMINARY ASSUMPTIONS.

#### Symbols Introduced

- $X$  . . . . Horizontal range.  
 $X'$  . . . . Inclined range.  
 $x$  . . . . Abscissa of any point in the trajectory.  
 $x_s$  . . . . Abscissa of the summit.  
 $y$  . . . . Ordinate of any point in the trajectory.  
 $y_s$  . . . . Ordinate of the summit, or maximum ordinate.  
 $D$  . . . . Drift.  
 $p$  . . . . Angle of position.  
 $\phi$  . . . . Angle of departure.  
 $\phi'$  . . . . Angle of elevation. (*sight angle*) + *between bore & sights*  
 $j$  . . . . Angle of jump.  
 $\omega$  . . . . Angle of fall.  
 $\theta$  . . . . Angle of inclination at any point in the trajectory.  
 $v$  . . . . Remaining velocity at any point in the trajectory.  
 $V$  . . . . Initial velocity (Also I.V.).  
 $v_\omega$  . . . . Striking velocity.  
 $v_s$  . . . . Summital velocity.  
 $v_h$  . . . . Horizontal velocity at any point in the trajectory.  
 $v_v$  . . . . Vertical velocity at any point in the trajectory.  
 $t$  . . . . Time of flight to any point in the trajectory.  
 $T$  . . . . Time of flight to point of fall.  
 $t_s$  . . . . Time of flight to the summit.

201. Ballistics is the science of the motion of projectiles. It is divided into two branches, namely, interior ballistics and exterior ballistics. Interior ballistics is that branch of the science which treats of the motion of the projectile while in the gun and of the phenomena which cause and attend this motion. Exterior ballistics treats of the motion of the projectile after it leaves the gun. The subject of interior ballistics is covered in the current edition of *Naval Ordnance* and will not be dealt with further here.

The path described by a projectile in flight is called the *trajectory*. Although the term *trajectory* is sometimes also applied to the path of a projectile through a medium other than air (for example, we speak of the under-water trajectory in dealing with the projectile's path through the water after impact on the surface of the sea), the term is commonly confined to the path of the projectile from the muzzle of the gun to the first point of impact, and we shall deal with it here in that sense. The following definitions pertain to elements of the trajectory. All of the angles defined are to be considered as measured in the vertical plane.

202. The *line of position* is the straight line joining the gun and target ( $OP$ , Figure 1).

The *angle of position* ( $p$ ) is the angle between the horizontal plane and the line of position; it is positive when the target is higher than the gun, and negative when the target is lower than the gun.

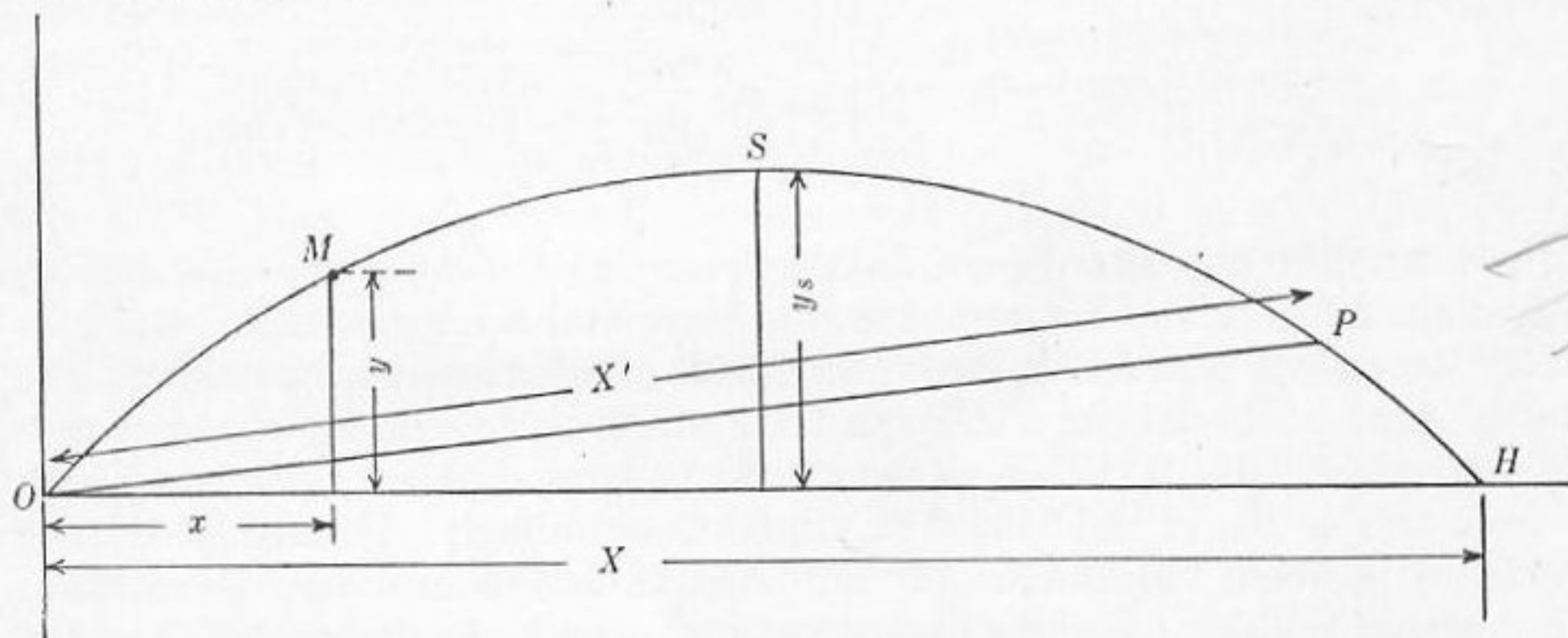
203. The *point of fall*, unless otherwise qualified, is the point at which the descending branch of the trajectory intersects the horizontal plane through the origin. For a target not in the latter plane, the point of fall may be taken as the point beyond the origin at which the trajectory intersects the line of position.

204. The distance in a straight line from the origin to a point of fall in the horizontal plane through the origin, is called the *horizontal range* ( $OH$ , Figure 1); it is denoted by the symbol  $X$ .

The distance in a straight line from the origin to a point of fall *not* in the horizontal plane through the origin, is called the *inclined range* ( $OP$ , Figure 1); it is denoted by the symbol  $X'$ .

As we shall ordinarily deal with the case of a point of fall in the horizontal plane through the origin, the term *range* will be used for convenience to denote horizontal range as defined above; the distinction provided for above will be made only when necessary.

The abscissa of any point in the trajectory (other than the point of fall in the horizontal plane through the origin) is denoted by the symbol  $x$ ; for the summit it is denoted by  $x_s$ .



$X$  = Horizontal range  
 $X'$  = Inclined range  
 $x$  = Abscissa of point  $M$

$y$  = Ordinate of point  $M$   
 $y_s$  = Maximum ordinate

FIGURE 1

205. The *summit*, or vertex, of the trajectory is the point at which the projectile ceases to ascend and commences to descend. If an actual trajectory should terminate in the ascending branch (as may occur in antiaircraft fire), the summit, in the sense here intended, will lie on an imaginary extension forward of the real trajectory. Similarly, if a gun should be fired downward so that the trajectory has only a descending branch, the summit will lie on an imaginary extension backward of the real trajectory.

The ordinate at the summit of the trajectory is called the *maximum ordinate*; it is denoted by the symbol  $y_s$ .

The ordinate at any other point in the trajectory is denoted by the symbol  $y$ .

206. The *line of departure* is the line in which the projectile is moving at the instant of projection; it is tangent to the trajectory at the origin.

The *angle of departure* ( $\phi$ ) is the angle between the horizontal plane and the line of departure ( $BOH$ , Figure 2).

The *angle of elevation* ( $\phi'$ ) is the angle between the line of position and the line of departure ( $BOP$ , Figure 2).

The *angle of jump* ( $j$ ) is the angle described by the axis of the bore, under the shock of firing, during the interval from the ignition of the charge to the ejection of the projectile from the muzzle ( $BOA$ , Figure 2); it is positive when the muzzle of the gun jumps upward, and negative when the muzzle of the gun jumps downward. In well-designed, modern guns the angle of jump amounts only to a few minutes of arc and remains reasonably constant, for a given gun, for all angles of departure; it is generally positive, but not invariably so.

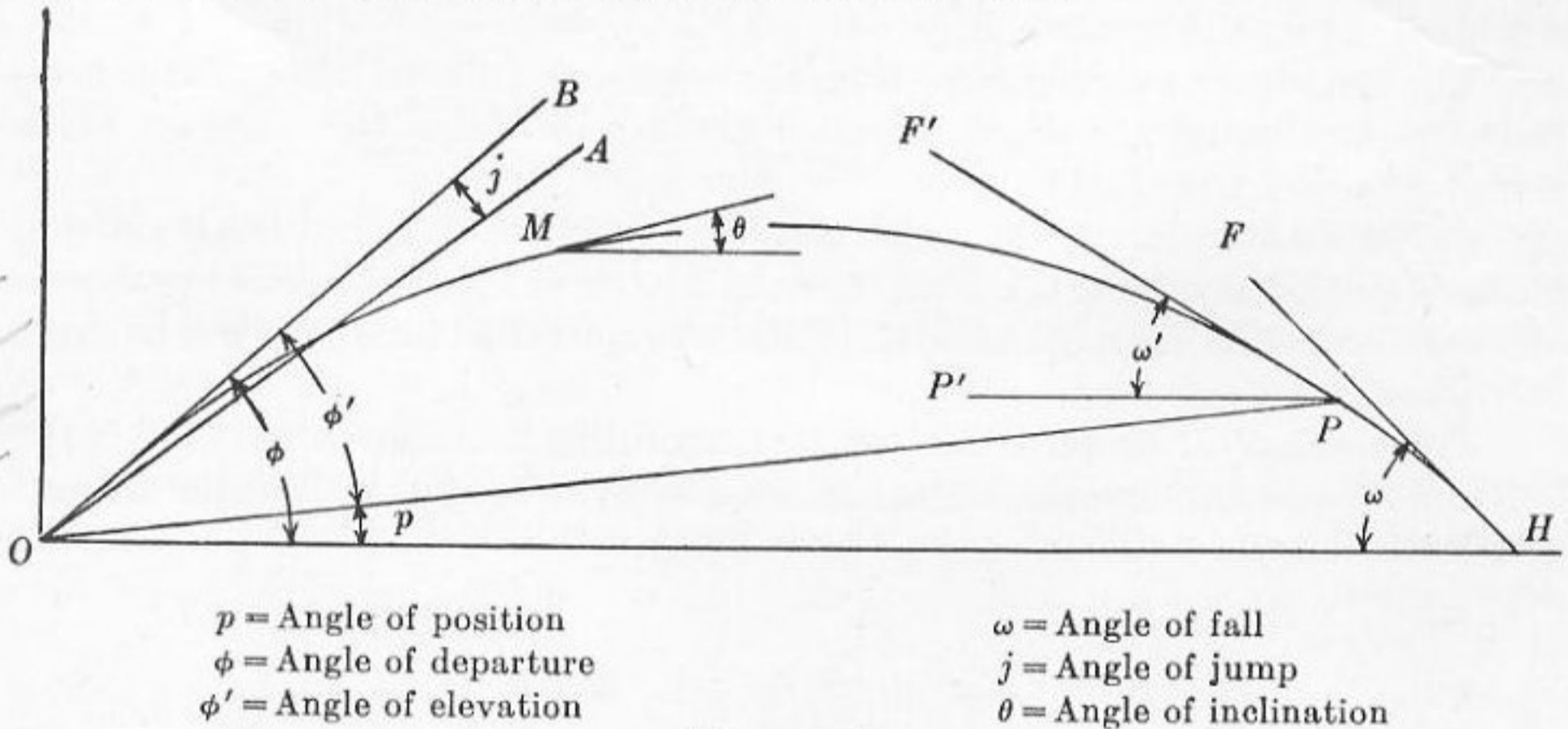


FIGURE 2

It is well to note here the relation between angle of departure and angle of elevation. When establishment of the true horizontal is impracticable, as is generally the case on board ship, some other plane of reference must be chosen. The line of sight to the target, which can be established by sighting on the target, affords such a plane. If the line of sight is not horizontal, the angle to be set with respect to the line of sight must be adjusted accordingly. The line of sight is evidently identical with the line of position as already defined above, and hence the angle of position measures the inclination between the line of sight and the horizontal. The same result will be obtained with the gun elevated the angle  $\phi' = \phi - p$  above the line of sight as will be obtained with gun elevated the angle  $\phi$  above the horizontal. The angle  $\phi'$ , as here defined, is then the angle at which the gun-sight telescopes must be set with respect to the bore, in order to give the gun the desired angle of elevation above the line of position (or sight). Strictly speaking, the angle to be set on the sights, and hence the angle of elevation, differs from the angle at which projection will actually occur, by the amount of the angle of jump. Or, in other words, the angle of elevation, in the sense that it measures the elevation of the bore above the line of sight just before firing, is really  $\phi' - j$ . However, the angle of jump is sufficiently small and uniform to allow it to be handled as an inherent part of the angle of elevation, and no distinction need be made between the angle of elevation and the angle of projection. This will be further clarified in article 804.

207. The *angle of fall* ( $\omega$ ) is the angle between the horizontal plane and the tangent to the trajectory at the point of fall ( $FHO$ , Figure 2). It is to be noted that the angle of fall is always measured with respect to the horizontal plane,

even when the point of fall is not in the horizontal plane through the origin. Thus at the elevated target,  $P$ , in Figure 2, the angle of fall is  $\omega' = F'PP'$ .

The *angle of inclination* ( $\theta$ ) at any point in the trajectory is the angle between the horizontal plane and the tangent to the trajectory at the given point. The angle of departure ( $\phi$ ) and angle of fall ( $\omega$ ) are, of course, special values of the angle of inclination applying, respectively, to the origin and point of fall.

208. The *initial velocity* ( $V$ ) is the velocity with which the projectile is supposed to leave the muzzle of the gun (the abbreviation I.V. is frequently used). The term *muzzle velocity* (abbreviated M.V.) is synonymous with initial velocity, but the latter term is generally used in exterior ballistics.

Since it is impracticable, for reasons that are readily apparent, to obtain a direct measure of the velocity of a projectile at the instant it leaves the muzzle, it is the practice to measure the velocity at as short a distance from the muzzle as practicable and to deduce therefrom the initial velocity. The process by which this is accomplished will be gone into later (arts. 806–808). It may be noted here, however, that the process employed assumes that the projectile has suffered normal retardation during its flight from the muzzle to the point of measurement. Actually, immediately after leaving the muzzle, the projectile is surrounded by a blast of gases moving more rapidly than the projectile, and experiments have shown that the projectile is accelerated during a travel of as much as fifty yards from the muzzle. Thus the value of initial velocity deduced in the manner just outlined is in fact a fictitious value; it is more accurately defined as the equivalent initial velocity, assuming projection into still air, that would produce the remaining velocity actually measured at a short distance from the muzzle.

209. The *remaining velocity* ( $v$ ) at any point in the trajectory is the velocity at that point measured in the tangent to the trajectory at that point.

The *striking velocity* ( $v_\omega$ ) is the remaining velocity at the point of fall.

The *summital velocity* ( $v_s$ ) is the remaining velocity at the summit, or vertex, of the trajectory.

The *horizontal velocity* ( $v_h$ ) at any point in the trajectory is the horizontal component of the remaining velocity at that point.

The *vertical velocity* ( $v_v$ ) at any point in the trajectory is the vertical component of the remaining velocity at that point.

210. The *time of flight*, when denoted by the symbol  $T$ , is the total time that elapses in the flight of the projectile from the gun to the point of fall. The time of flight to any other point in the trajectory is denoted by the symbol  $t$ .

211. The *drift* ( $D$ ) is the perpendicular distance of the point of fall from the vertical plane containing the line of departure ( $HH'$ , Figure 3).

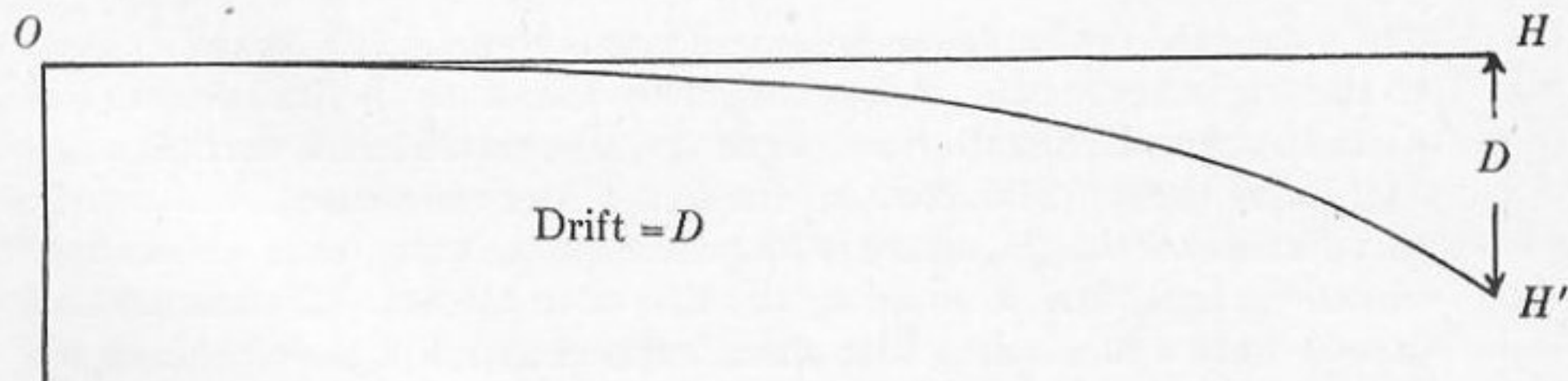


FIGURE 3

212. The units in which values of the various elements of the trajectory are expressed must be given careful consideration in numerical problems, since

current practice as to the use of units is by no means uniform. In the range tables of the U.S. Navy the following is standard practice:

- Units
- (a) Velocities are expressed in foot-seconds.
  - (b) Horizontal distances are expressed in yards.
  - (c) Vertical distances are expressed in feet.
  - (d) Angles are expressed in sexagesimal units.

Exercises in this book ordinarily will develop solutions in these units.

It is to be noted, however, that many of the sources of ballistic data used by us have originated outside of our own service, and usually are in metric units. In these sources all distances are commonly expressed in meters, and velocities in meter-seconds. In order to avoid the multiplicity of symbols that would result from adopting separate symbols for the several elements to denote the different units in which they may be expressed, no distinction as to units will be attached to symbols. Due caution must be exercised, therefore, to state the units applicable whenever numerical values are involved.

**Preliminary assumptions** 213. Before proceeding with the problem of solving the trajectory the following preliminary assumptions will be made:

1. The dimensions of the gun are negligible in comparison with the trajectory; for convenience we shall assume the origin of the trajectory to be at the intersection of the bore with the trunnion axis of the gun, and the trunnion axis to be horizontal and located at the surface of the earth.
2. All positions will be referred to a system of rectangular axes, with the origin coinciding with the origin of the trajectory as defined above; and with the  $X$ -axis horizontal, perpendicular to the trunnion axis, and positive in the direction of fire; the  $Y$ -axis vertical and positive upward; and the  $Z$ -axis horizontal, perpendicular to the  $X$ -axis, and positive to the right with respect to the direction of fire. This results in the establishment of a datum plane tangent to the earth at the origin, and this system of coordinates in exterior ballistics is therefore referred to as the *tangent-plane system*. The fact that the surface of the earth departs from this tangent plane may be accounted for by very simple corrections whenever necessary, as will be brought out later.\*
3. The force of gravity acts always in a direction perpendicular to the datum plane established above.\*
4. The earth and its atmosphere and the gun itself all are motionless. These assumptions are made only for the purpose of obtaining a first approximation of the motion of the projectile. Later we shall take into account separately the effects of the earth's motion, and of wind and motion of the gun.
5. The axis of the projectile remains at all points coincident with the tangent to the trajectory. Under this assumption, since both the initial force imparted by the gun, and the force of gravity, are exerted in the vertical plane containing the line of departure, the force of air resistance is confined to this plane, and the trajectory is therefore a plane curve confined entirely to the vertical plane containing the line of departure. This assumption is made only while dealing with a first approximation of the motion of the

\* In some works on exterior ballistics positions are referred to a curved system which may be thought of as the usual Cartesian grid bent to conform to the surface of the earth. This system has no material advantages over the tangent-plane system, and, as will appear later (article 813), the latter system actually is the one more applicable to problems in naval gunnery.

projectile. Later we shall take into account separately the effects of forces not satisfying this assumption.

The above assumptions are general in their application and serve principally to strip the problem of certain complications which can more readily be dealt with after the principal effects have been accounted for. Additional assumptions will be introduced at appropriate points later in the text.

## EXERCISES

1. For the following angles of elevation and position, what are the corresponding angles of departure?

Problem	DATA				ANSWERS	
	Angle of elevation		Angle of position		Angle of departure	
1.....	2°	00'	+15°	00'	+17°	00'
2.....	3	00	+12	15	+15	15
3.....	3	00	-10	30	-7	30
4.....	2	00	-12	07	-10	07
5.....	3	00	+11	15	+14	15
6.....	5	00	+10	16	+15	16
7.....	4	00	-9	37	-5	37
8.....	6	00	-6	22	-0	22

2. A target is at a horizontal distance of 3000 yards from the gun, and is 750 feet higher than the gun above the water. Compute the angle of position.

*Answer.*  $p = 4^{\circ} 45' 49''$ .

3. A target is at a horizontal distance of 10,000 yards from the gun, and is on the water 1500 feet below the level of the gun, the latter being in a battery on a hill. Compute the angle of position.

*Answer.*  $p = (-)2^{\circ} 51' 45''$ .

4. A target is at a horizontal distance of 1924 yards from the gun, and is 1123 feet higher above the water than the gun. Find the angle of position and the distance in a straight line from the gun to the target in yards.

*Answers.*  $p = 11^{\circ} 00' 35''$ .  $X' = 1960$  yards.

5. A target is at a horizontal distance of 1860 yards from the gun, and it is on the water 1238 feet below the level of the gun, the latter being in a battery on a hill. Find the angle of position and the distance in a straight line from the gun to the target in yards.

*Answers.*  $p = (-)12^{\circ} 30' 33''$ .  $X' = 1905$  yards.

THE EQUATION OF THE TRAJECTORY IN VACUUM.  
THE THEORY OF RIGIDITY OF THE TRAJECTORY.

301. Although the resistance of the atmosphere to the motion of projectiles traveling at high velocities is sufficiently great to render any calculations which neglect such resistance worthless for giving information for actual firing, certain similarities exist between the trajectory in vacuum and the trajectory in air; these similarities pertain rather to general characteristics of the trajectory than to any quantitative values of the elements. Since the solution of the trajectory in vacuum is very simple, while the solution under conditions of air resistance becomes very complex, it is desirable to make a preliminary study of the general characteristics of the trajectory from a study of the curve in vacuum. From this study it will also be possible to make some deductions that will have an important application to practical problems dealing with the trajectory in air.

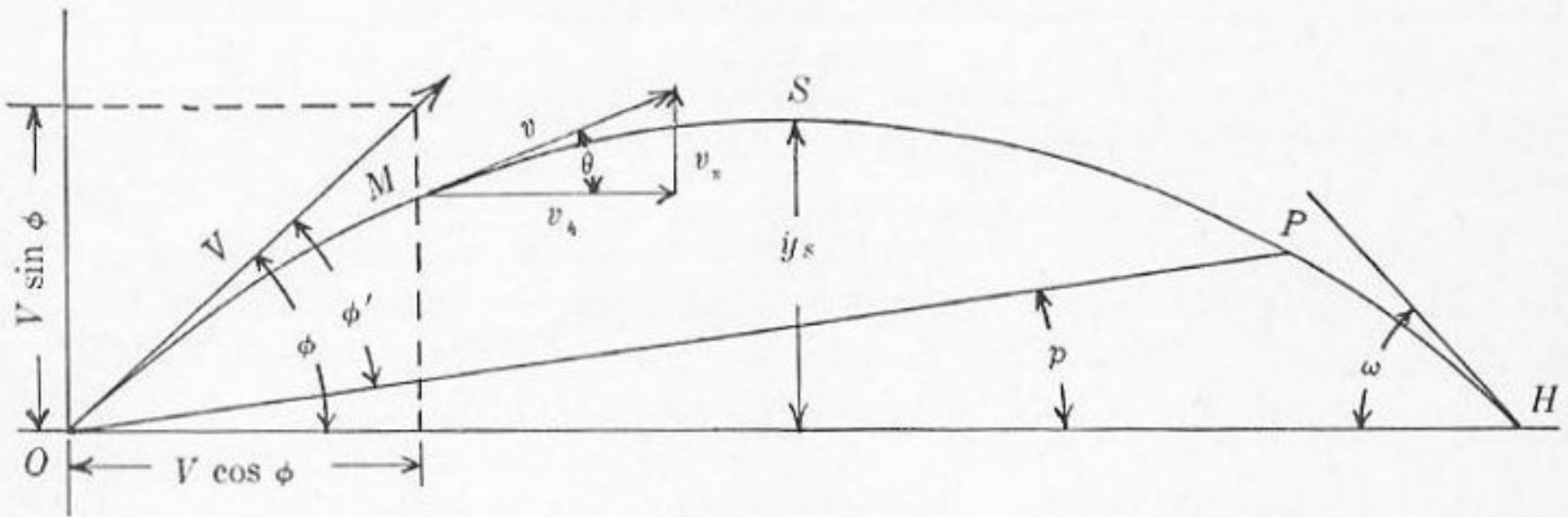


FIGURE 4

302. In Figure 4, the curve  $OSH$  represents the trajectory in vacuum, with origin at  $O$  and point of fall at  $H$  in the horizontal plane through  $O$ . The line marked  $V$  is the line of departure, its length representing the initial velocity  $V$ ; the horizontal and vertical components of the latter,  $V \cos \phi$  and  $V \sin \phi$ , respectively, are also shown.

Since the force of gravity is exerted only in the vertical plane, the horizontal distance ( $x$ ) covered by the projectile in any interval of time ( $t$ ) is the horizontal component of the initial velocity multiplied by the interval of time, or

$$x = t V \cos \phi. \quad (301)$$

In the vertical plane the projectile's travel is influenced by gravity as well as by the vertical component of its velocity. The vertical distance ( $y$ ) covered by the projectile in any interval of time ( $t$ ) is then the vertical component of the initial velocity multiplied by the interval of time, less the distance the projectile falls due to the force of gravity in the same interval of time, or

$$y = t V \sin \phi - \frac{1}{2}gt^2. \quad (302)$$

The above equations are called the *primary equations*, as from them all other deductions are made.

## Elements at any point

303. Substituting for  $t$  in (302) its value as found from (301) we have

Equation of  
the trajectory  
in vacuum

$$y = x \tan \phi - \frac{gx^2}{2V^2 \cos^2 \phi} \quad (303)$$

This is the equation of the trajectory in vacuum, and its form indicates that the trajectory is a parabola with vertical axis.

304. Since  $\frac{dy}{dx} = \tan \theta$ , we have, by differentiating (303),

Angle of  
inclination  
at any point

$$\tan \theta = \tan \phi - \frac{gx}{V^2 \cos^2 \phi} \quad (304)$$

which gives an expression for the *angle of inclination* at any point in the trajectory.

305. By differentiating equations (301) and (302) with respect to time we obtain, respectively, the horizontal and vertical components of the remaining velocity at any time ( $t$ ).

$$\frac{dx}{dt} = V \cos \phi; \quad \frac{dy}{dt} = V \sin \phi - gt.$$

By combining these components we get the remaining velocity at any point in the trajectory.

$$v^2 = \left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2 = (V^2 \sin^2 \phi - 2gt V \sin \phi + g^2 t^2) + V^2 \cos^2 \phi$$

and since  $\sin^2 \phi + \cos^2 \phi = 1$ , the above reduces to

$$v^2 = V^2 - 2gt V \sin \phi + g^2 t^2 = V^2 - 2g \left( t V \sin \phi - \frac{gt^2}{2} \right).$$

The expression in the bracket being the value of  $y$  from (302), we have finally

Remaining  
velocity at  
any point

$$v = \sqrt{V^2 - 2gy}. \quad (305)$$

Equation (305) shows that the remaining velocity is equal to the initial velocity when  $y=0$ , which occurs both at the origin and at the point of fall. Also, the remaining velocity has its smallest value where  $y$  has its greatest value, or at the summit; it follows that the velocity must decrease up to the summit and then increase until at the point of fall it has again reached its initial value.

306. The horizontal and vertical velocities at any point in the trajectory are, by definition, the horizontal and vertical components, respectively, of the remaining velocity at that point, whence

Horizontal  
and vertical  
velocities at  
any point

$$v_h = v \cos \theta; \quad v_v = v \sin \theta. \quad (306)$$

It is to be observed, however, that in a vacuum no force operates to diminish the initial value of the horizontal velocity, since the only force operating on the projectile, other than that initially imparted by the gun, is gravity, and the latter acts only in the vertical plane. Hence the horizontal velocity at any point in the tra-

jectory must be equal to the horizontal component of the initial velocity, or  $v_h = V \cos \phi$ .

307. The time of flight to any point in the trajectory is obtained directly from (301), and is

Time of  
flight to  
any point

$$t = \frac{x}{V \cos \phi}. \quad (307)$$

#### Elements at the point of fall

308. The value of  $x$  at the point where the trajectory again cuts the horizontal plane through the origin is the *horizontal range*. Since at this point  $y=0$ , we may obtain an expression for the horizontal range by substituting  $y=0$  in (303). Factoring the resulting expression, we find

$$x \left( \tan \phi - \frac{gx}{2V^2 \cos^2 \phi} \right) = 0.$$

This gives for one value  $x=0$ , which is evidently the origin. For the other we may now use the appropriate symbol for the horizontal range.

$$\begin{aligned} \frac{gX}{2V^2 \cos^2 \phi} &= \tan \phi \\ X &= \frac{2V^2 \cos^2 \phi \tan \phi}{g} \end{aligned} \quad (308A)$$

which simplifies as follows

$$X = \frac{2V^2 \cos^2 \phi \sin \phi}{g \cos \phi} = \frac{V^2(2 \sin \phi \cos \phi)}{g}$$

Horizontal  
range

$$X = \frac{V^2 \sin 2\phi}{g}. \quad (308)$$

Equation (308) shows that the horizontal range increases with the angle of departure until the latter reaches  $45^\circ$ , while for further increases in the angle of departure the range decreases; the same range is given by either of two angles of departure, one as much greater than  $45^\circ$  as the other is less than  $45^\circ$  (for example, the same range is given by  $\phi = 60^\circ$  as by  $\phi = 30^\circ$ ). Also, for a given angle of departure the range varies as the square of the initial velocity.

Equation (308) may also be re-written

$$\sin 2\phi = \frac{gX}{V^2} \quad (308B)$$

in which form it provides an expression for the angle of departure required to obtain a given range. If the right-hand member is greater than unity it indicates that the given range is not obtainable with the given initial velocity.

309. By putting for  $x$  in (304) its value  $X$  for the point of fall, we may find the angle of inclination at that point, or the *angle of fall*. For convenience the expression for  $X$  from (308A) may be taken; using in place of  $\theta$  the symbol  $\omega$  for angle of fall, we now have

$$\tan \omega = \tan \phi - \frac{2V^2 \cos^2 \phi \tan \phi g}{V^2 \cos^2 \phi g} = \tan \phi - 2 \tan \phi$$

whence

$$\text{Angle of fall} \quad \tan \omega = -\tan \phi. \quad (309)$$

This shows that the angle of fall is numerically equal to the angle of departure.

310. By substituting  $y=0$  in (305) we obtain values of the remaining velocity for both origin and point of fall. The former is, of course, the initial velocity; the latter is the *striking velocity*, and (305) shows that the striking velocity equals the initial velocity.

$$\text{Striking velocity} \quad v_{\omega} = V. \quad (310)$$

311. The *time of flight* to the point of fall is obtained by putting  $X$  for  $x$  in (307).

$$\text{Time of flight to point of fall} \quad T = \frac{X}{V \cos \phi}. \quad (311)$$

#### Elements at the summit

312. By definition, the summit of the trajectory is the point at which the projectile ceases to ascend and commences to descend. At this point the projectile must then momentarily be in horizontal flight, and the angle of inclination ( $\theta$ ) of the curve at this point must be zero. Substituting  $\theta=0$  in (304), and denoting the abscissa of this point by its special symbol  $x_s$ , we have

$$0 = \tan \phi - \frac{gx_s}{V^2 \cos^2 \phi}$$

whence

$$x_s = \frac{V^2 \cos^2 \phi \tan \phi}{g}$$

Simplifying this expression in the manner already shown for (308A), and comparing the result with (308), we have

$$\text{Horizontal distance to summit} \quad x_s = \frac{V^2 \sin 2\phi}{2g} = \frac{X}{2}. \quad (312)$$

This indicates that the summit is at mid-range.

313. By substituting in (303) the value of  $x_s$  just found, we find the ordinate for the summit, or *maximum ordinate* ( $y_s$ ).

$$y_s = \frac{V^2 \cos^2 \phi \tan^2 \phi}{g} - \frac{gV^4 \cos^4 \phi \tan^2 \phi}{2g^2 V^2 \cos^2 \phi}$$

which simplifies directly to

$$y_s = \frac{V^2 \sin^2 \phi}{2g}. \quad (313A)$$

The above may further be simplified by multiplying both numerator and denominator of the right-hand term by  $\cos \phi$ , after which this term becomes divisible by the expression for  $X$  from (308) and finally reduces to

$$\text{Maximum ordinate} \quad y_s = \frac{X \tan \phi}{4}. \quad (313B)$$

A very convenient expression for the maximum ordinate in terms of time of flight is arrived at from the consideration that the height of the summit equals the height from which the projectile would fall in one-half the time of flight ( $T$ ).<sup>\*</sup> This results directly in the expression

$$y_s = \frac{1}{2}g \left( \frac{T}{2} \right)^2 = \frac{gT^2}{8}. \quad (313C)$$

314. The time of flight to the summit is obtained by substituting the value of  $x_s$  for  $x$  in (307); it is, however, equally convenient to note that since  $x_s = \frac{X}{2}$

Time of  
flight to  
summit

$$t_s = \frac{T}{2}. \quad (314)$$

315. By putting for  $y$  in (305) the value of the maximum ordinate ( $y_s$ ), an expression for the remaining velocity at the summit may be obtained. However, as has already been noted in article 312, the projectile is momentarily in exactly horizontal flight at the summit, hence the remaining velocity at the summit equals the horizontal velocity at that point. And since, as noted in article 306, the horizontal velocity at any point equals the horizontal component of the initial velocity, we have directly

Remaining  
velocity at  
summit

$$v_s = V \cos \phi. \quad (315)$$

316. The deductions made above may now be summarized as follows: They all pertain to the trajectory in vacuum, but in some instances they remain approximately true also for the trajectory in air, as noted herein.

General  
characteris-  
tics of the  
trajectory  
in vacuum

1. The trajectory is a parabola with vertical axis; this is approximately true also in air.

2. At the origin the angle of inclination is identical with the angle of departure; at the summit it is zero; at the point of fall it is identical with the angle of fall and equal numerically to the angle of departure.

For the origin and summit these conclusions are equally true for the trajectory in air; the angle of fall in air is, however, materially greater than the angle of departure.

3. The remaining velocity decreases from its initial value until the summit is reached, and then increases until at the point of fall it becomes again equal to the initial velocity, i.e., the striking velocity equals the initial velocity. In air the remaining velocity also decreases in the ascending branch and increase in the descending branch, but the increase starts at a point beyond the summit; in air the striking velocity is always very materially less than the initial velocity.

4. The horizontal velocity remains constant throughout the trajectory, while the vertical velocity decreases in the ascending branch and becomes zero at the vertex and in the descending branch increases until at the point of fall it is equal (but opposite in direction) to its value at the origin. In air the horizontal velocity is constantly reduced; the vertical velocity vanishes at the summit and increases in the descending branch, as in vacuum, but does not again attain its initial value.

<sup>\*</sup> This deduction may be verified by equating the values of  $y_s$  given by (313A) and (313C) and solving for  $T$ . The resulting value of  $T$  may then be equated to the value of  $T$  from (311), and this equation in turn solved for  $X$ . The final result is an expression for  $X$  identical with (308).

5. The range increases with increase of angle of departure up to a maximum value of  $45^\circ$  for the latter, but for further increase in the angle of departure the range decreases. Also, the same range is given by two angles of departure, one of which is as much greater than  $45^\circ$  as the other is less than  $45^\circ$ . The first of these conclusions is approximately true also in air although, for reasons to be brought out later, with modern guns, projectiles, and velocities, the limiting angle is somewhat greater than  $45^\circ$  (about  $50^\circ$  for large guns). The second conclusion is true in air to the extent that for angles of departure greater than the limiting value (for air), the range is decreased, but the relation that has been stated for vacuum conditions holds only very approximately in air.

6. For a given angle of departure, the range varies as the square of the initial velocity. This is approximately true in air; actually, in air, range increases somewhat less than in proportion to the square of the initial velocity.

7. The summit, or vertex, of the trajectory is located at mid-range, and the time of flight to the summit is one-half the time of flight to the point of fall. These relations remain approximately true in air. For large guns the range to the summit, in air, varies from about .51 to about .53 of the total range, while for small guns the range to the summit may become almost .60 of the total range. The time of flight to the summit, in air, is generally between .46 and .49 of the total time of flight, and usually nearer the latter figure.

### The theory of rigidity of the trajectory

317. Since in actual practice it usually occurs that the target is not in the horizontal plane through the gun, it is of interest to extend our study of the trajectory in vacuum to a comparison between a trajectory referred to an inclined plane and one referred to the horizontal plane. To do this we will seek a relation between the inclined range  $X'$  resulting from an angle of elevation  $\phi'$  and angle of position  $p$ , and the horizontal range  $X$  resulting from an angle of departure  $\phi$  equal to  $\phi'$ .

In Figure 4 let  $x, y$  represent the coördinates of the point  $P$  on the trajectory  $OPH$ , and  $X'$  the inclined range to that point. The angle of elevation with respect to the line of position  $OP$  is  $\phi' = \phi - p$ , and the angle of position is defined

by  $\tan p = \frac{y}{x}$ . Equation (303) may be written

$$\frac{y}{x} = \tan \phi - \frac{gx}{2V^2 \cos^2 \phi}$$

whence

$$\tan p = \tan \phi - \frac{gx}{2V^2 \cos^2 \phi}$$

and

$$x = \frac{2V^2}{g} \cos^2 \phi (\tan \phi - \tan p).$$

The right-hand term of this expression may be expanded to

$$\tan \phi - \tan p = \frac{\sin \phi \cos p - \cos \phi \sin p}{\cos \phi \cos p} = \frac{\sin (\phi - p)}{\cos \phi \cos p}$$

whence,

$$x = \frac{2V^2 \sin(\phi - p) \cos \phi}{g \cos p} = \frac{2V^2 \sin \phi' \cos(\phi' + p)}{g \cos p}.$$

Since  $X' = x \sec p$ , we have finally

$$X' = \frac{2V^2 \sin \phi' \cos(\phi' + p)}{g \cos^2 p}. \quad (316)$$

Equation (316) is an expression for the inclined range  $X'$  resulting from an angle of elevation  $\phi'$  with respect to a line of position making the angle  $p$  with the horizontal. Equation (308) is an expression for the horizontal range  $X$  resulting from an angle of departure  $\phi$ . In order to compare the horizontal range with the inclined range for the condition  $\phi = \phi'$ , let us rewrite (308) with  $\phi'$  in place of  $\phi$ , and expand the term  $\sin 2\phi'$ .

$$X = \frac{2V^2 \sin \phi' \cos \phi'}{g}.$$

Dividing (316) by the above we have

$$\frac{X'}{X} = \frac{\cos(\phi' + p)}{\cos \phi' \cos^2 p}.$$

After expanding  $\cos(\phi' + p) = \cos \phi' \cos p - \sin \phi' \sin p$ , the above reduces to

$$\frac{X'}{X} = \sec p(1 - \tan \phi' \tan p). \quad (317)$$

318. The right-hand term of (317) becomes equal to unity when  $p = 0$ , which is to be expected since in that case  $X'$  and  $X$  are identical. If  $p$  is positive, the right-hand term has a value less than unity, which means that  $X'$  is less than  $X$ ; this is also to be expected since this situation represents the case of firing *up* an incline. Similarly, for the case of firing *down* an incline,  $X'$  becomes greater than  $X$ .\*

But as long as  $p$  remains very small  $\sec p$  does not vary much from unity and  $\tan p$  does not vary much from zero; under these conditions the entire right-hand term of (317) does not vary much from unity. In naval gunnery we have ordinarily to deal only with very small angles of position, except in the case of anti-aircraft fire. In surface fire it would be most unusual to encounter an angle of position greater than about  $35'$ , and a value as great as this occurs only at extremely short ranges, in which case  $\phi'$  is also small. For example, for a  $3''$  gun mounted 50 feet above the water line and firing at the water line of a target 1600 yards distant, the angle of position is about  $(- )35'$  and the angle of elevation  $48'$ . The ratio

$\frac{X'}{X}$  in this case is 1.0001 and the variation quite negligible. For the same gun, at

5000 yards with  $p$  now reduced to  $(- )11'$  and  $\phi'$  increased to  $5^\circ 17'$ , the ratio becomes 1.0003; at 8500 yards, with  $p = (- )7'$  and  $\phi' = 15^\circ 35'$ , the ratio becomes 1.0006. These examples represent rather extreme situations for fire from ships against surface targets. Under ordinary conditions the ratio will not vary appreciably from unity.

\* These conclusions are also generally true in air as long as the angle of position does not become large. With large angles of position the reverse may become true. Ref. §38, *Handbook of Ballistics*, Vol. I, by Cranz and Becker.

319. We may conclude from this investigation that with very small angles of position the range is not affected appreciably by the angle of position. This means that we may use the same angle of elevation whether the target is on the same level as the gun or on a different level, provided that we limit ourselves to such cases as are encountered in ordinary practice, *excluding antiaircraft fire*.

The assumption just made is known as the *assumption of the rigidity of the trajectory*. It means, in effect, that a given trajectory may be swung through a small vertical angle, as illustrated in Figure 5, without appreciably altering its form. It is important to note, however, that in any comparison of elements of trajectories under the terms of the above assumption, the elements must be referred

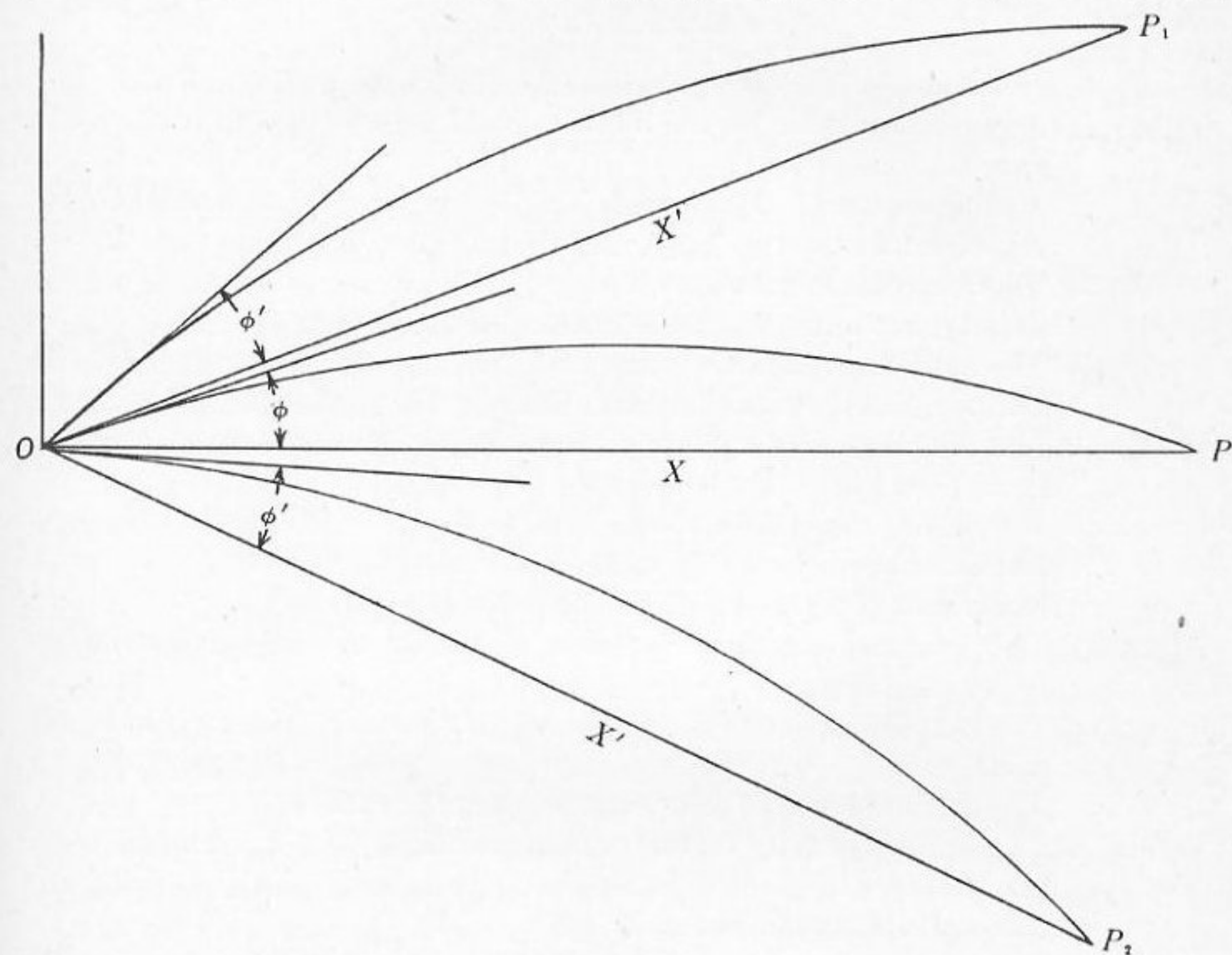


FIGURE 5

to their respective lines of position. For example, the maximum ordinates of the three curves in Figure 5 may be considered equal only if each is measured perpendicularly from the mid-range point of its own line of position; according to the usual definition of maximum ordinate, the values of this element are, of course, affected appreciably even by small angles of position. This applies to any elements which ordinarily are measured with respect to the horizontal plane. The immediate application of the assumption of rigidity of the trajectory, in actual service, is the use of the same sight-bar graduations for firing at targets which may be either slightly elevated or slightly depressed with respect to the gun, such as a gun on a high deck firing at a submarine, or a submarine's gun firing at the superstructure of a ship.

Although the foregoing assumption has here been based only on an investigation of conditions in vacuum, it is equally applicable in air, for the effects of air resistance on the quantities  $X$  and  $X'$  are so nearly equal as to cause no appreciable alteration in the ratio between these quantities.

THE RETARDATION OF THE PROJECTILE DUE TO RESISTANCE  
OF THE ATMOSPHERE. THE COEFFICIENT OF FORM.  
THE BALLISTIC COEFFICIENT.

## New Symbols Introduced

- $R_f$  . . . . Resistance of the air against the flight of a projectile (in lbs.).
- $R_a$  . . . . A general notation for the retardation of a projectile due to air resistance (in f.s.s.).
- $A$  . . . . A constant used in Mayevski's retardation functions (art. 409).
- $a$  . . . . An exponent used in Mayevski's retardation functions (art. 409).
- $G_v$  . . . . The retardation of a standard projectile, in air of standard surface density, according to the  $G$ -function (in f.s.s.) (art. 411).
- $H_v$  . . . . The standard relation assumed for the variation of air density with altitude, based on standard air density at the surface (art. 420).
- $E$  . . . . The retardation of a given projectile in the standard atmosphere (in f.s.s.) (art. 429).
- $w$  . . . . Weight of projectile (in lbs.).
- $d$  . . . . Diameter of projectile (in inches).
- $i$  . . . . Coefficient of form of a projectile (arts. 413–417).
- $\delta_s$  . . . . The assumed standard for density of air at the earth's surface (1.2034 kg/m<sup>3</sup>).
- $\delta_a$  . . . . The actual density of air at the earth's surface, under given conditions.
- $\delta$  . . . . The surface density factor, representing the ratio  $\delta_a/\delta_s$ .
- $\delta_b$  . . . . The ballistic density factor, or ballistic density, for a given trajectory (art. 421).
- $C$  . . . . The ballistic coefficient (art. 428).
- $g$  . . . . The acceleration due to gravity (to be taken as 32.16 f.s.s. unless otherwise indicated).

401. Although the effect of air resistance was not altogether dismissed by earlier investigators, Newton appears to have been the first to deal quantitatively with this factor. From purely theoretical considerations he determined that the retardation due to the air resistance must vary in proportion to the square of the velocity, provided that the effects of pressure diminution behind the projectile and of the air stream set up by friction might be neglected. He tested this principle by dropping spheres of various densities from the dome of St. Paul's Cathedral, and found it to hold approximately true for the comparatively low velocities thus attained. But subsequent experiments soon showed that the effects of air resistance on the flight of projectiles cannot be accounted for by anything so simple as Newton's quadratic law.

402. Robins was the first to measure the velocity of projectiles with tolerable accuracy. In 1742 he published his famous *New Principles of Gunnery*, wherein he described his invention, the *ballistic pendulum*, and stated the results

**The ballistic pendulum** obtained therewith. The ballistic pendulum consisted of a heavy bob suspended from a tripod, arranged so as to receive the impact of a projectile and measure the resulting swing; from the velocity imparted to the bob, the striking velocity of the projectile was determined. With this device Robins measured velocities up to about 1700 f.s., and determined approximately the loss of velocity during flights up to about 250 feet from the gun. The results thus obtained were approximately in agreement with Newton's quadratic law for velocities up to about 900 f.s., but revealed great discrepancies between this law and observed results for higher velocities. Robins also suggested using the gun itself as a pendulum for determining the initial velocity, and Rumford soon thereafter experimented with such a device.

Hutton made extensive experiments in England from 1775 to 1791, using both the gun pendulum and the ballistic pendulum. In France, in 1839-40, the *Commission de Metz* conducted further extensive experiments with the ballistic pendulum, and from the results obtained established empirical formulas for the calculation of range and time of flight for various angles of departure.

**403.** Experiments with the ballistic pendulum ceased when, in 1840, Wheatstone suggested measuring the velocity of a projectile by causing it to cut successive wire screens, each of which formed part of an electric circuit containing an element for measuring time. **The chronograph** The *Commission de Metz* in 1856-57 undertook new experiments, using a chronograph designed by Navez, but the results did not have lasting significance. The experiments of an English clergyman, Bashforth, in 1866-70, using a chronograph designed by himself, were notable as being the first to lead to a fairly accurate analysis of the effects of air resistance. Bashforth not only measured air resistance with fair accuracy, but also showed how the resistance varied with different types of projectile head; among other things, he concluded that the resistance is proportional to the projectile's cross-sectional area. Many experimental programs followed soon after the development of the chronograph. The most important were: in France, the *Commission de Gâvre*, 1856-61, 1873, and 1888; in England, Bashforth, 1866-70, and 1878-80; in Russia, Mayevski, 1868-69; in Germany, Krupp, 1875-81; in Holland, Hojel, 1884. A most notable contribution in this period was the analysis made by Mayevski, and later extended by Zaboudski, of the results of the English, Russian, and German experiments.

Wheatstone's proposal that velocity might be measured by causing the projectile to interrupt electrical circuits, initiated the first real progress in the science of ballistics, and this principle has continued up to the present day to be an essential factor in the advancement of the science. The *Boulengé* chronograph became the most successful of instruments of its type and was used extensively until recently. The principle of interrupting circuits by causing the projectile to cut wire screens was later used with oscillographs. At the present time a variation of this principle is used, in the form of the so-called solenoid type of oscillograph, which depends upon electrical impulses generated as a magnetized projectile passes through wire loops placed in its path.\*

**404.** As an illustration of a method by means of which the retardation of a projectile may be measured with the aid of a chronograph (or its equivalent), the following elementary description and example will suffice.

The gun is fired practically horizontally through screens located at such dis-

\* For complete descriptions of the instruments here mentioned, see *Naval Ordnance*.

Experimental  
determination  
of retardation

tances from the muzzle that the portion of the trajectory intercepted by them is very nearly a horizontal, straight line. We may then consider, with negligible error, that the length of the trajectory between the points of measurement is equal to the horizontal

distance between these points, and that the retardation measured between these points is not affected by the force of gravity and hence is due only to air resistance. Two pairs of screens and two chronographs are required; the distance between each pair, and the distance between the midpoints of the two pairs, must be known accurately. A chronograph is placed in circuit with each pair of screens, and as the projectile passes through them the time of flight between the screens of each pair is measured. The average velocity between each pair of screens is then determined by dividing the distance between the screens by the measured time of flight; for each pair this velocity may be considered to apply to the midpoint of that pair. We shall denote the velocity at the first pair by  $v_1$  and at the second pair by  $v_2$ , and the loss of velocity between the two pairs (i.e., between their midpoints) by  $v_1 - v_2$ . The average retardation for the flight between the two midpoints is found by dividing the loss of velocity by the time elapsed between these points; the latter may be determined (without appreciable error for the short distance involved) by dividing the distance  $l$  between the two midpoints by the average velocity  $\frac{v_1 + v_2}{2}$  between these points. Then we have

finally,

$$R_\alpha = \frac{(v_1 - v_2)(v_1 + v_2)}{2l}. \quad (401)$$

The retardation thus determined applies to a velocity which is approximately the mean of the two measured, or  $\frac{v_1 + v_2}{2}$ \*

In order to determine the total air resistance against the projectile at the same average velocity, we may apply the law of physics,  $F = \frac{w}{g} \alpha$ . We shall let  $R_f$  denote the force of air resistance (in lbs.).  $R_\alpha$  as already determined is the acceleration (or deceleration in this case);  $w$  is the weight of the projectile. Then

$$R_f = \frac{w}{g} R_\alpha. \quad (402)$$

An application of the above formulas is given in the following example.

*Given:* A 6" projectile weighing 105 lbs. was fired practically horizontally through two pairs of screens. The distance from the gun to the midpoint of the first pair was 400 feet, and to the midpoint of the second pair 915 feet. The velocity, as measured by chronograph, was 2221 f.s. between the first pair and 2173 f.s. between the second pair.

---

\* By firing through several pairs of screens the rate of change of retardation may be determined and the correspondence between retardation and velocity more accurately defined. A complete treatment of this problem is given on pp. 51-59, *New Methods in Exterior Ballistics*, by F. R. Moulton.

The sharply pointed wave front shown in these pictures may be accounted for as follows. In Figure 6 let the line  $AE$  represent the line of flight of a projectile, and  $E$  be the position of the tip of the projectile at the instant depicted. At  $E$  the tip of the projectile is just impacting on the air at that point, and the resulting disturbance will travel as a spherical wave of condensation centered at  $E$  (as a stone thrown into water sets up a circular ripple). The same thing has already happened at earlier points in the flight of the projectile, as at  $A$ ,  $B$ ,  $C$ , and  $D$ , and we therefore find each of these points now the center of a wave spreading

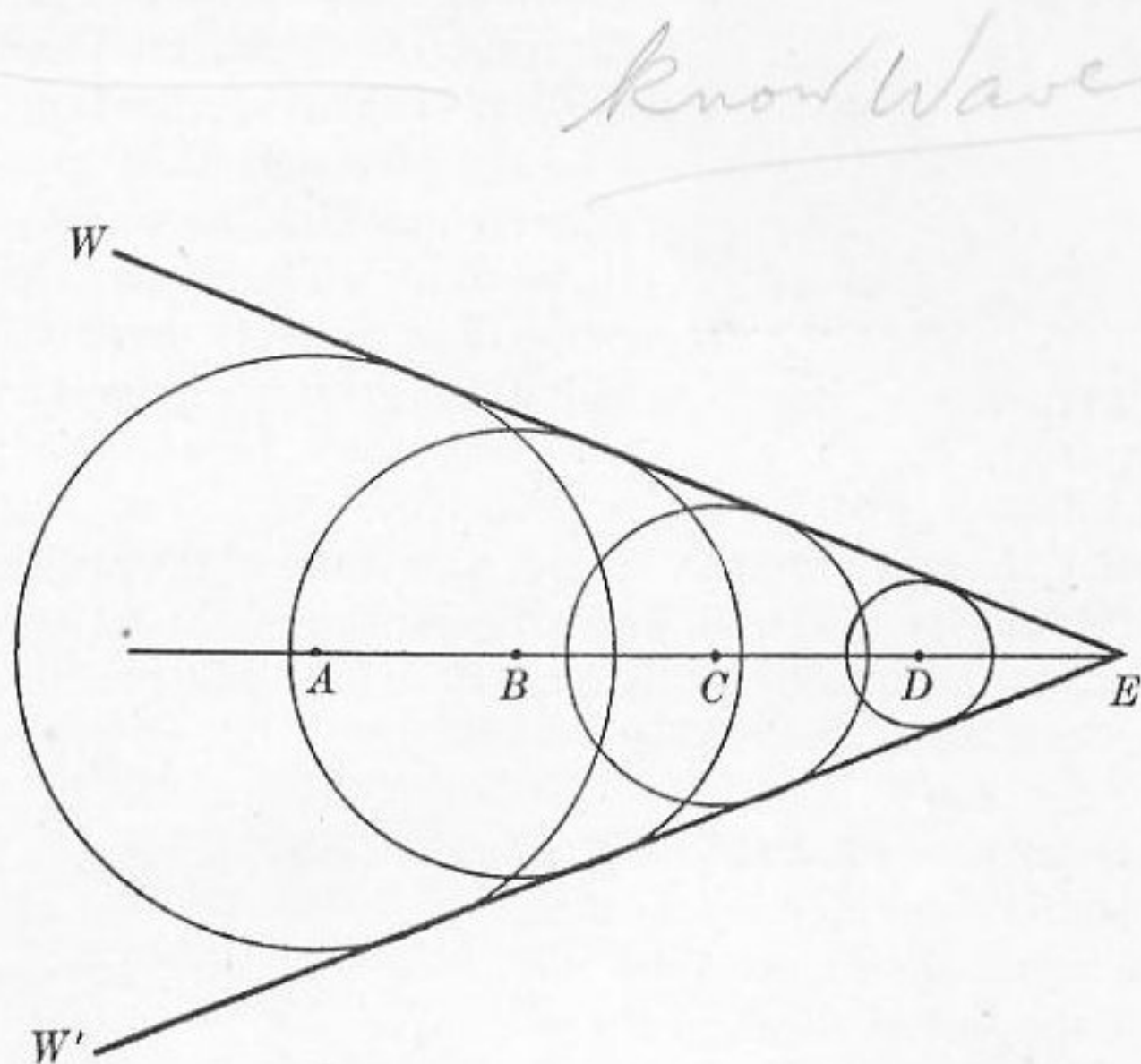


FIGURE 6

outward. The waves are receding from their respective initial points of disturbance at the velocity of sound, and the radii at the several points are proportional to the intervals of time elapsed since the projectile passed them. Under the assumed construction, a line drawn from  $E$  tangent to any one of these spheres is tangent to all of them, as shown by  $EW$  and  $EW'$ . Considering an infinitely great number of successive positions of the projectile between  $A$  and  $E$ , the peripheries of the resulting waves evidently form solid lines  $EW$  and  $EW'$ , and these lines then form the wave front shown in the photographs. In the upper photograph of Plate I the wave front of the tail wave is seen to be an envelope for such circular waves. A particularly vivid demonstration of this feature is shown in Plate II. This is a photograph of the air disturbance attending the flight of a bullet (approximately ".30 cal., at 2900 f.s., as in Plate I) through a perforated

*Find:* What was the average retardation between the points of measurement, and to what velocity does it apply (approximately)? What was the total resistance encountered by the projectile at this velocity?

For substitution in (401),  $l = 915 - 400 = 515$  feet,  $v_1 + v_2 = 4394$  f.s., and  $v_1 - v_2 = 48$  f.s.

$(v_1 - v_2) = 48$ . . . . .	log 1.68124
$(v_1 + v_2) = 4394$ . . . . .	log 3.64286
$2l = 1030$ . . . . .	log 3.01284 . . . . .
	<u>colog 6.98716-10</u>
$R_\alpha = 204.76$ f.s.s. . . . .	log 2.31126

With formula (402) we find,

$R_\alpha = 204.76$ . . . . .	log 2.31126
$w = 105$ . . . . .	log 2.02119
$g = 32.16$ . . . . .	log 1.50732 . . . . .
	<u>colog 8.49268-10</u>
$R_f = 668.54$ lbs. . . . .	log 2.82513

The retardation of 204.76 f.s.s. and resistance of 668.54 lbs. apply to the mean velocity 2197 f.s. (approximately).

405. The nature of the results obtained in the various experiments to measure the retardation due to air resistance soon indicated that the factors entering into this retardation are exceedingly complex. Practically all of the earlier attempts to reconcile the observed results with theory clung to the quadratic law as a general basis and sought to bring about agreement, under the terms of a quadratic law, by supplying the latter with coefficients varying for different velocity bands. Even this expedient failed to yield satisfactory agreement and it eventually was accepted that retardation varies according to a variable power of the velocity. An examination of the phenomena attending the flight of a projectile will serve to explain some general features of the laws that eventually were developed.

406. Photography of projectiles in flight has been an invaluable aid in revealing some of the most important of these phenomena. With suitable lighting arrangements photography can be made to show very clearly the nature of the disturbances of the air medium surrounding a projectile in flight. Plate I\* shows three photographs of an 8 mm. bullet (approximately ".30 cal.) in flight through air at about 2900 f.s.

The conditions for all three of these photographs were the same, the differences in the appearance of the air disturbance being caused by differences in lighting. The remarkably clear revealment of the waves issuing from the head and tail, and of the turbulence to the rear of the projectile, is brought about by refraction of light through the condensations and rarefactions of the air at these points.

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\* Plates I, II, and III, have been reproduced from *Handbook of Ballistics*, Volume I, Cranz and Becker, by courtesy of the publisher (Julius Springer, Berlin). For additional photographic records, and description of details of making the records, see the above reference; also, *Spark Photography and Its Application to Problems in Ballistics*, by Philip P. Quayle (Scientific Papers of the Bureau of Standards, No. 508); also, *A Camera for Studying Projectiles in Flight*, by H. L. Curtis, W. H. Wadleigh, and A. H. Sellman (Technological Papers of the Bureau of Standards, No. 255); also *Lehrbuch der Ballistik*, Volume III, Cranz (5th Edition).

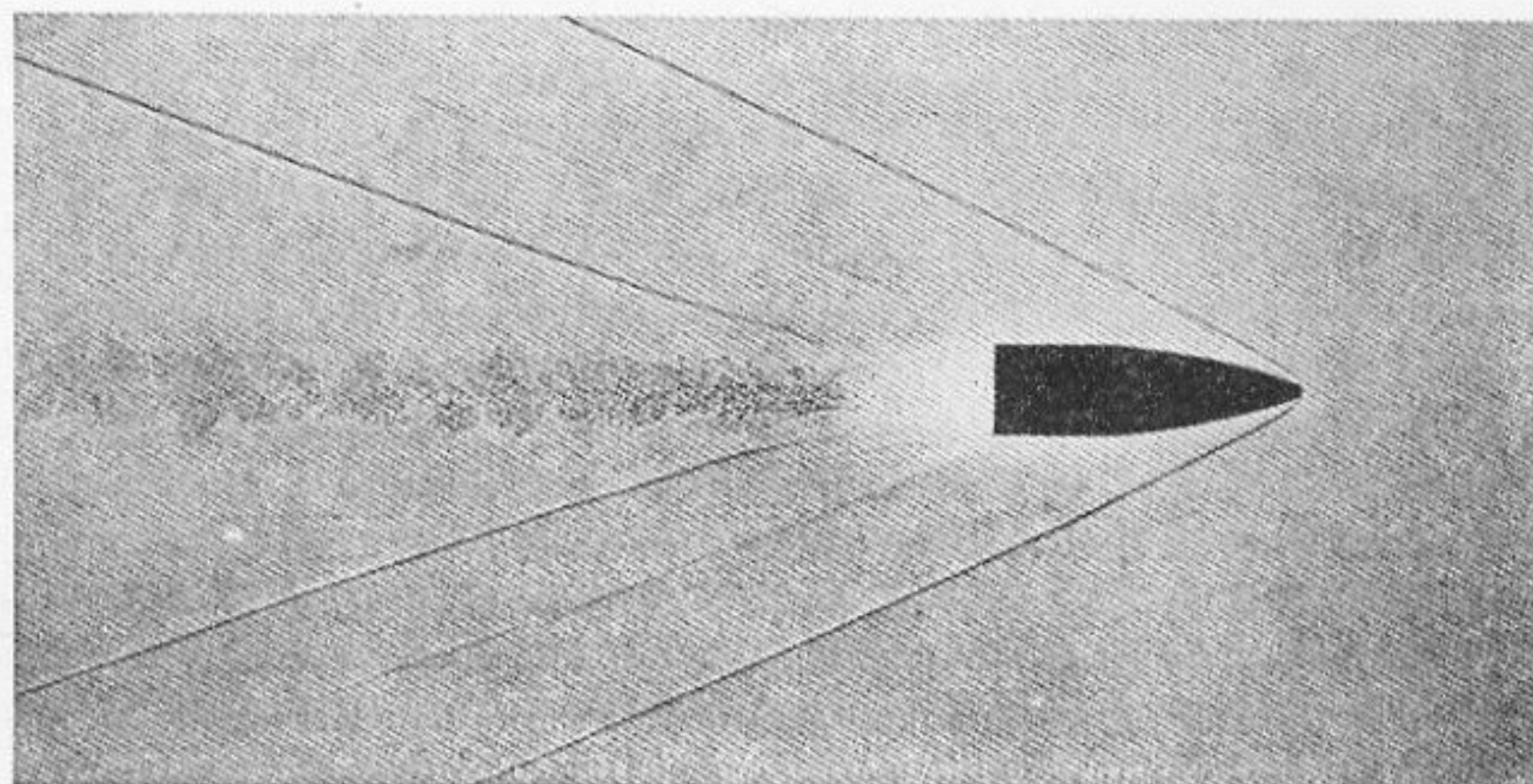
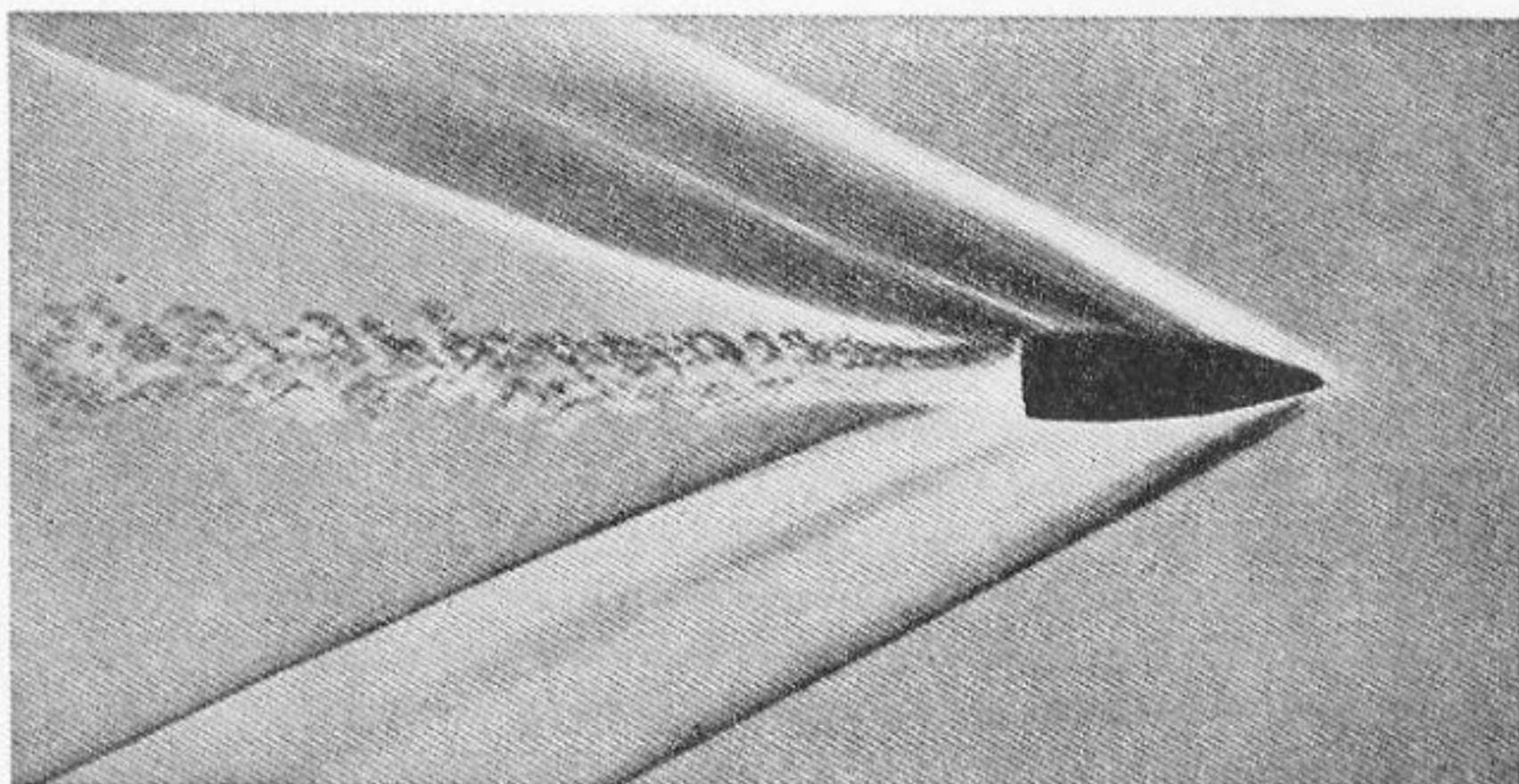
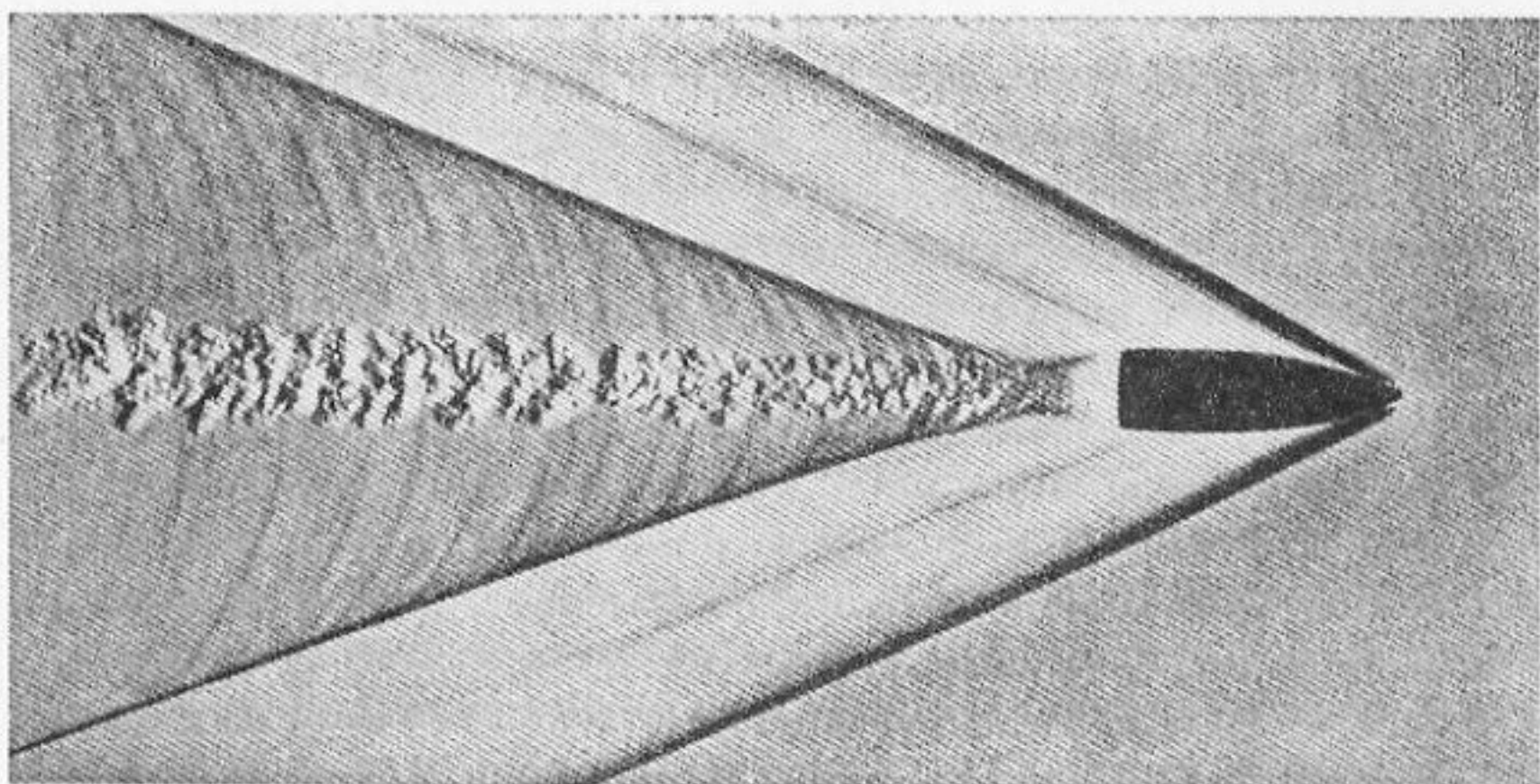


PLATE I

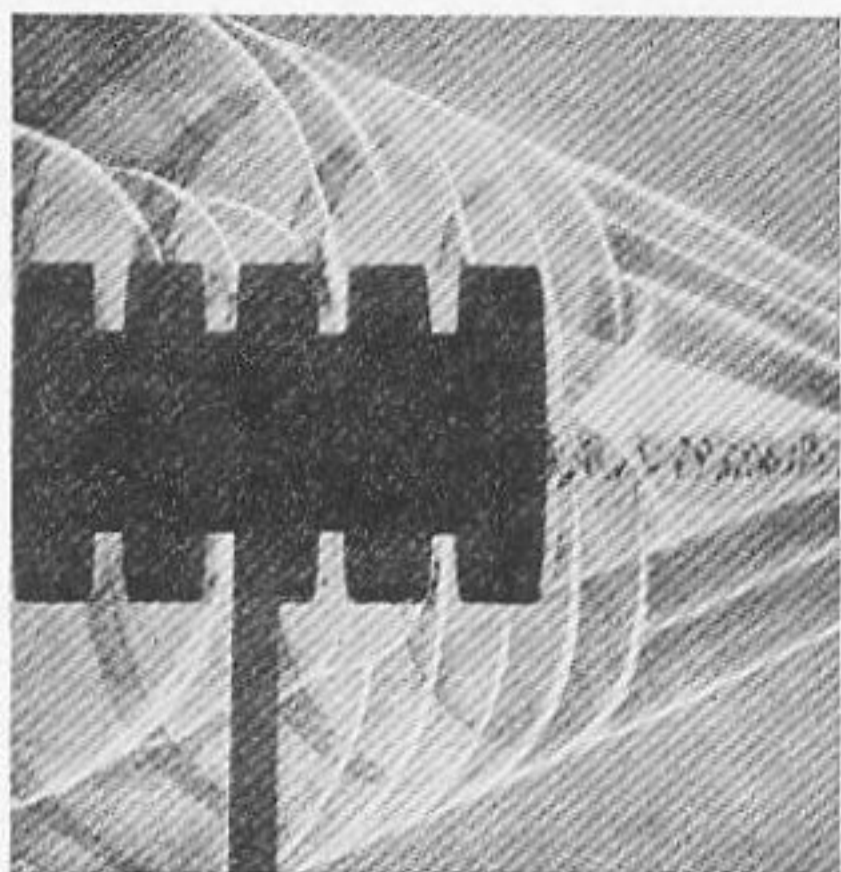


PLATE II

waves' outward motion, and hence the more acute will be the angle of the wave front. The slope of the wave front is, indeed, a measure of the projectile's velocity compared to the velocity of sound, but irregularities of the latter under the abnormal conditions in the immediate vicinity of the projectile prevent it from being used as an accurate measure in this connection.

Let us consider now what should happen if the projectile's velocity is less than the velocity of sound. In Figure 7 let  $AE$  again represent the line of flight of a projectile, but in this case a projectile travelling at a velocity less than that of sound.

By the time the projectile has reached point  $E$ , the wave initiated by it at point  $A$  will have preceded it to  $A_1A_2$ , and similarly the waves initiated at  $B$ ,  $C$ , and  $D$  will have preceded it as shown. Plate III also illustrates this case; it is a photograph of a bullet travelling at about 1115 f.s., which, under the attending conditions, was somewhat less than the velocity of sound. Here the wave front has disappeared and we see, instead, a detached wave of condensation in advance of the projectile.

408. From theoretical considerations, as well as from the indications obtained photographically, it appears that the loss of energy of a projectile in flight is due to the following effects. No attempt is made here, however, to evaluate the proportionate amounts of these effects.

tube. The purpose of this arrangement was to separate the waves, by permitting only a portion to escape through the holes in the tube. By suitable lighting effects these escaping waves were then shown, as they emerged from the tube, combining to form the wave front as before.

407. The shape of the wave fronts shown in Plates I and II and in Figure 6 depends, of course, on the fact that the velocity of the projectile is greater than the velocity of sound (i.e., greater than the velocity at which the waves of disturbance travel in air). The greater the excess of the projectile's velocity over the velocity of sound, the greater will be the ratio between the projectile's forward motion and the

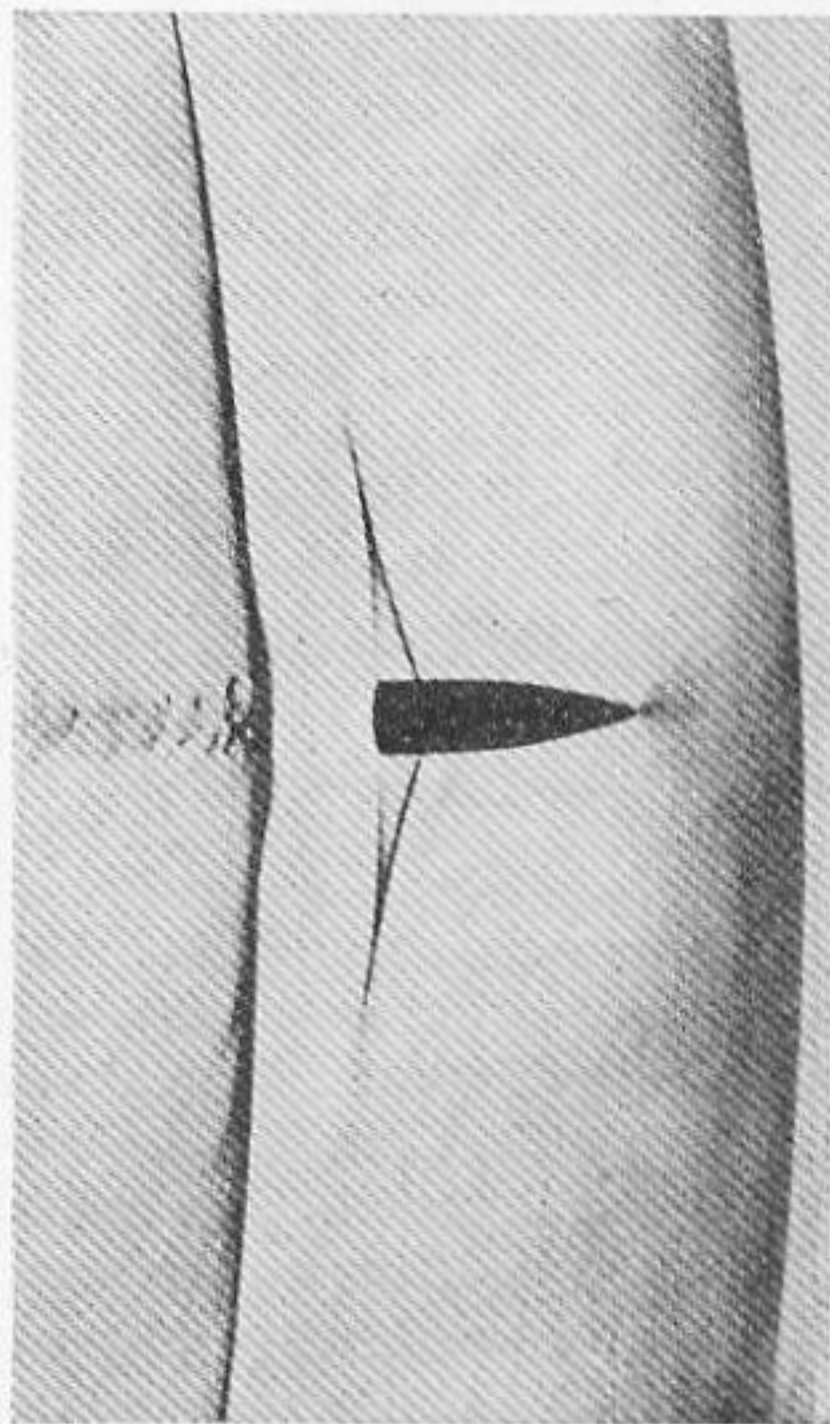


PLATE III

1. Energy is dissipated in the creation of air waves. The amount of this effect evidently is influenced by the form of the projectile, as well as by the area of its cross section.
2. Energy is dissipated in the creation of suction and eddy currents; the marked turbulence in the wake of the projectile (Plates I, II, and III) gives some evidence as to the degree of this effect. This effect evidently is influenced chiefly by the form of the projectile, and particularly by the form of its after-body.
3. Energy is dissipated in the form of heat generated by frictional resistance. This effect evidently is influenced by the form of the projectile, as well as by the area and character of its surface.

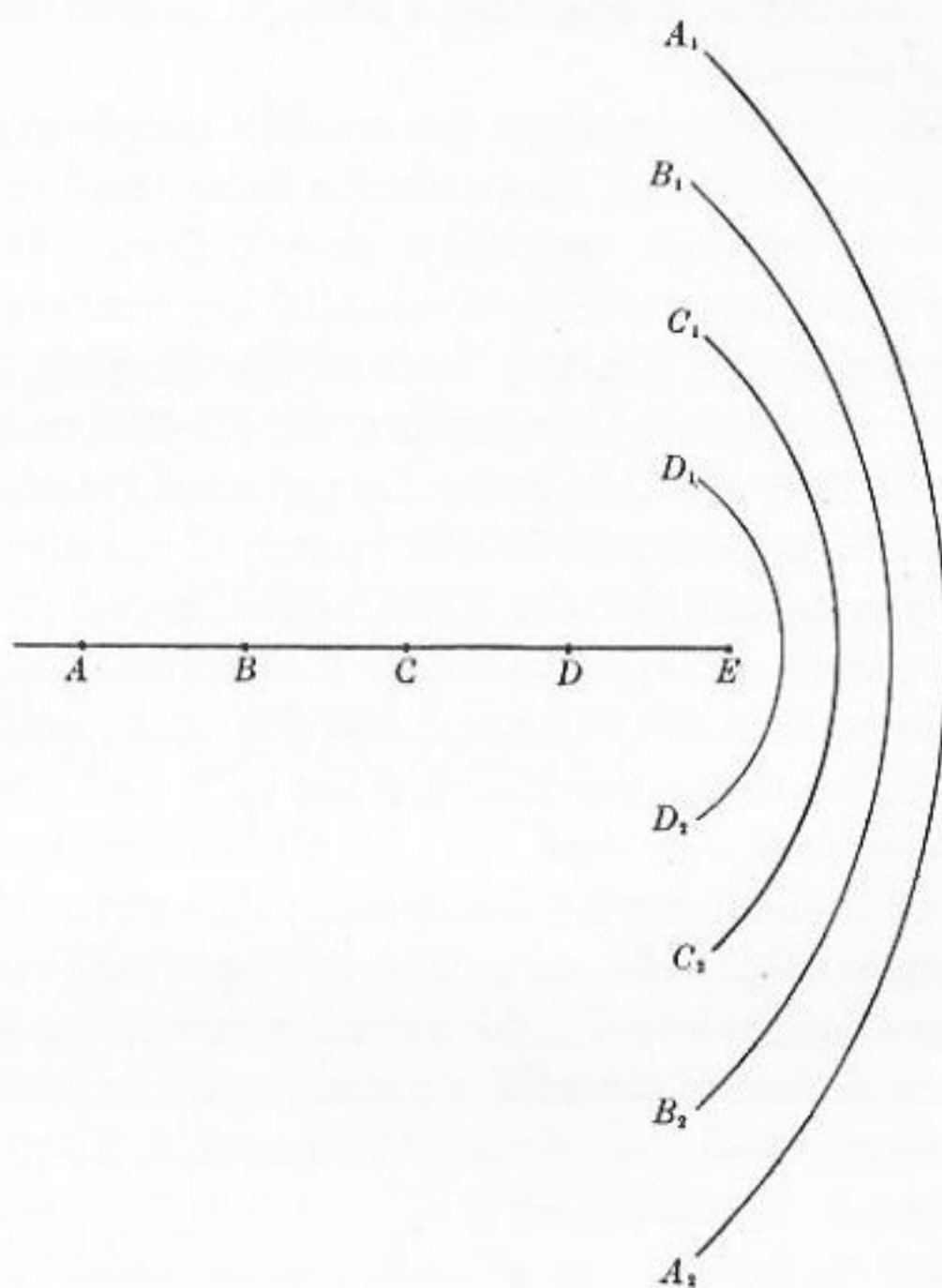


FIGURE 7

Although sufficient knowledge to permit separate evaluation of these several effects is not at hand, the indications are that the first two enumerated probably account for most of the total loss of energy suffered by the projectile. The loss of energy due to the creation of waves is minimized by giving the projectile a smooth contour, for points of sharp change in contour are sources of additional wave disturbances. Plates I and III show that the rotating band and the rear face of the projectile, as well as its tip, are sources of wave disturbances. The suction and eddy effects are influenced materially by the shape of the projectile's after-body. Considering only the problem of minimizing air resistance, the shape of the projectile should be very much like that of an airship. However, other considerations prevent anything more than an approximation of such a shape; the need

for rotating bands and for uniform passage through the bore calls for a cylindrical body and prevents use of the streamline shape.\*

409. Consideration of the wave phenomena revealed by photographic records, and of their explanations as outlined above, must lead at once to the conclusion that the relation between velocity of projectile and velocity of sound has a most important bearing on the resistance offered by the air to a projectile's flight. Furthermore, considering the very marked change in the condition of affairs according to whether the projectile's velocity is above or below the velocity of sound, it is very reasonable to suppose that any law which might be set up to account for observed results must be capable of showing a very marked, perhaps even abrupt, change at the velocity of sound. Indeed, one might even suspect that perhaps one law should be sought for velocities less than the velocity of sound and a totally different one for velocities greater than the velocity of sound.

In the search for an adequate and yet reasonably simple expression of a retardation law, quadratic, cubic, and bi-quadratic laws, and combinations of these, were tried, but none of these remained long in favor. Mayevski appears to have been the first to introduce the idea of establishing zones of velocities, with different exponents for each zone. On the basis of the English and Russian experiments of 1866-70 he established three zones, viz., 0-280 m.s., 280-360 m.s., and 360-510 m.s.; for the first of these zones he assumed resistance to vary according to a combination of square and fourth power of velocity, for the second according to the sixth power, and for the third according to the square. Hojel, from his own experiments of 1884, established five zones, viz., 140-300 m.s., 300-350 m.s., 350-400 m.s., 400-500 m.s., and 500-700 m.s., and for these zones he assumed, respectively, the exponents 2.5, 5, 3.83, 1.77, and 1.91.

The most widely accepted series of laws, or functions, was that given by Mayevski in 1881, and extended later by Zaboudski; this series of functions was based on the Krupp firings of 1875-81, as well as on Mayevski's own experiments of 1868-69 and Bashforth's of 1866-70. Mayevski's functions have until quite recently been accepted as standard for ballistic computations in our own services, and they are even at present used by us to a limited extent. They are represented by the general expression

$$R_{\alpha} = \frac{A}{C} v^a \quad (403)$$

in which  $R_{\alpha}$  denotes the retardation corresponding to the velocity  $v$ , and the coefficient  $A$  and exponent  $a$  vary from zone to zone but remain constant within a given zone. The ballistic coefficient,  $C$ , combines factors pertaining to the weight, cross-section, and form of the projectile, and to the atmospheric density, all of which will be discussed presently. However, it is appropriate to note here that the standard of form assumed by Mayevski was that of a projectile about three calibers long with an ogival head rounded to a two-caliber radius. Mayevski's functions, as translated into English units by Col. Ingalls (U. S. Army), are given below.

\* A slight tapering at the rear end of the projectile, known as "boat-tailing," has been applied successfully to projectiles up to about 6" caliber.

$v$  between 3600 f.s. and 2600 f.s.

$$R_\alpha = \frac{A_1}{C} v^{1.55} \quad \log A_1 = 7.60905 - 10$$

$v$  between 2600 f.s. and 1800 f.s.

$$R_\alpha = \frac{A_2}{C} v^{1.7} \quad \log A_2 = 7.09620 - 10$$

$v$  between 1800 f.s. and 1370 f.s.

$$R_\alpha = \frac{A_3}{C} v^2 \quad \log A_3 = 6.11926 - 10$$

$v$  between 1370 f.s. and 1230 f.s.

$$R_\alpha = \frac{A_4}{C} v^3 \quad \log A_4 = 2.98090 - 10$$

$v$  between 1230 f.s. and 970 f.s.

$$R_\alpha = \frac{A_5}{C} v^5 \quad \log A_5 = 6.80187 - 20$$

$v$  between 970 f.s. and 790 f.s.

$$R_\alpha = \frac{A_6}{C} v^3 \quad \log A_6 = 2.77344 - 10$$

$v$  between 790 f.s. and 0 f.s.

$$R_\alpha = \frac{A_7}{C} v^2 \quad \log A_7 = 5.66989 - 10$$

Mayevski's  
retardation  
functions

410. In recent years it has become the practice to abandon all formal expressions of a velocity-retardation relation, or series of such expressions, and to resort to explicit use of a tabulated relation. The latter practice minimizes compromise between experiment and theory, insofar as the velocity-retardation relation is concerned, for it makes practically direct use of observed data. Reduction of observed results to a convenient system of standard conditions represents, of course, the application of theory to some degree. Any tabulated function may be expressed also by a single analytical equation, by making the latter as complicated as necessary. In the case of the velocity-retardation function, such an expression becomes exceedingly complicated and totally worthless for any purposes of direct application to solutions of the trajectory. Such expressions have been developed, but their application is confined to extrapolation beyond the limits of observed values, in tabulating the function.\*

The results obtained by the *Commission de Gâvre* in its experimental program of 1888 were reduced to a tabular velocity-retardation function that came to be known as the *Gâvre function* and to be accepted extensively in the United

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\* A single analytical expression closely approximating Mayevski's functions was set up by Siacci, but it contained some 17 factors in complicated arrangement; this expression is given on page IV of *Artillery Circular M* (1917), U. S. Army. The single expression representing the Gâvre function is of similar complexity; it may be found on page VI of *Exterior Ballistic Tables Based on Numerical Integration*, Volume I (U. S. Army, 1924). The latter reference states also the sources of the original Gâvre function and of its modifications to the form now in use.

The *G*-function States as well as abroad. In the United States a slightly modified version of the Gâvre function, referred to generally as the *G*-function, is used in connection with present-day computations of trajectories, and the numerical illustrations and exercises presented in this text will be confined to the use of the latter.

411. The *G*-function may be thought of as a tabular expression of the relation between retardation and velocity, that is, as a table from which may be found directly the retardation corresponding to any given velocity. Denoting retardation (according to the *G*-function) by  $G_v$ , and velocity by  $v$ , this amounts then simply to a tabulation of  $G_v$  against the argument  $v$ . We will refer to this tabulation as the *G*-table\* (Table I, *Range and Ballistic Tables, 1935*).

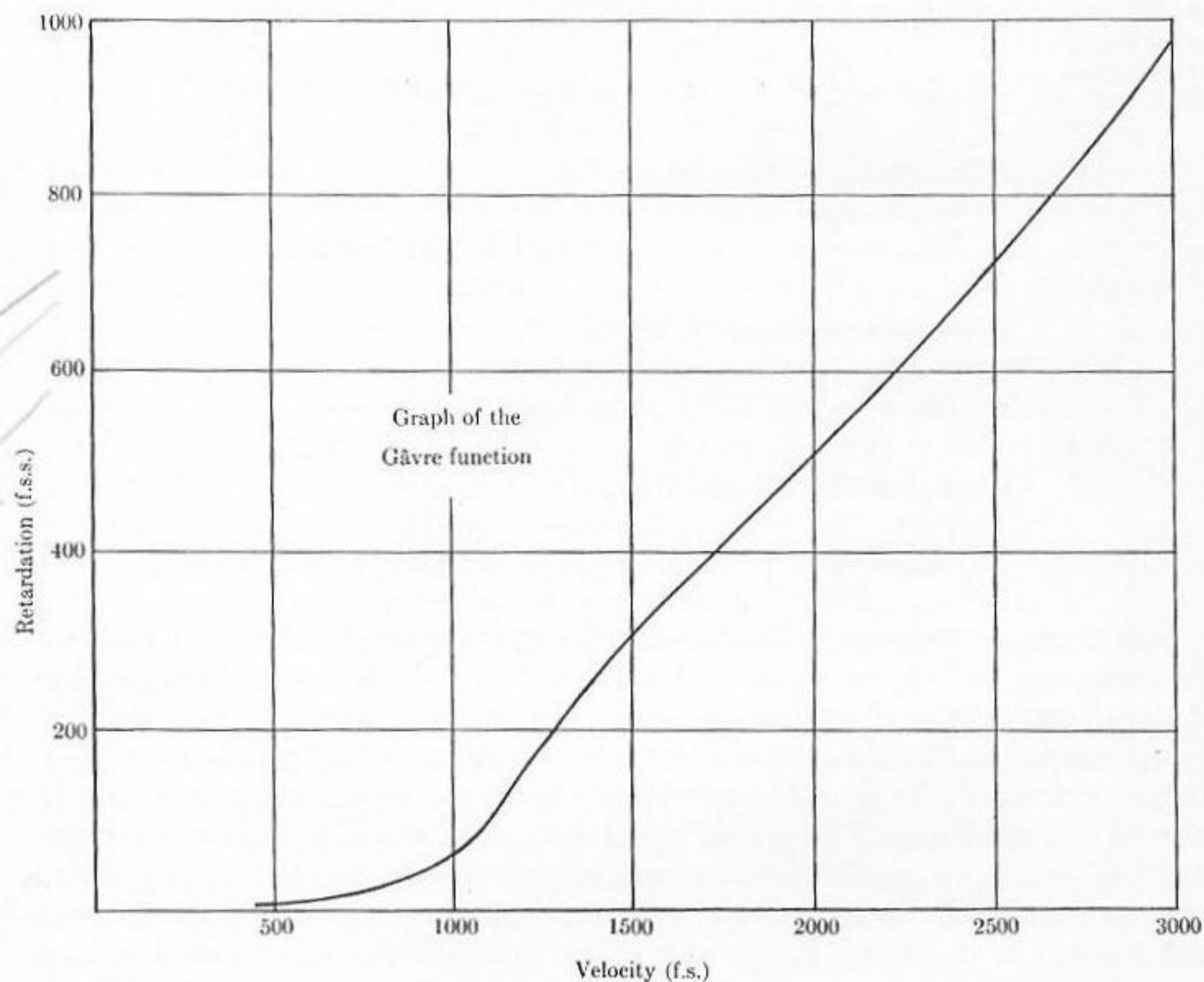


FIGURE 8

The graph of the *G*-function is shown in Figure 8. It is of particular interest to note the marked change in the character of the curve as it approaches the velocity of sound (about 1100 f.s.), and also the peculiar inflexions between about

\* Since in many methods actually in use for computing trajectories the expressions  $\frac{G_v}{v}$  or  $\frac{G_v}{v^2}$  occur frequently, tables of the *G*-function usually are in terms of these expressions rather than in terms of  $G_v$  directly (as here defined). Also the entering argument is usually  $\frac{v^2}{100}$ , for a similar reason. In all discussions and exercises in this text, however, the

term  $G_v$  is meant to denote retardation directly, and in the *G*-table employed it is tabulated against  $v$  directly.

1000 f.s. and about 1500 f.s. (compare this with the behavior of the exponent of Mayevski's functions in the same region). This curve then bears out what already has been suspected from an investigation of the photographic records of Plates I, II, and III. The graph of Mayevski's functions, although somewhat less continuous, agrees closely with the  $G$ -curve; the greatest discrepancy occurs around 2500 f.s., where it is slightly greater than 3% (in terms of the retardation argument). Recent experiments (1926-33) conducted by the U. S. Army, using several types of projectile varying materially among each other as well as from the type used in the Gâvre experiments, show even more remarkable results.

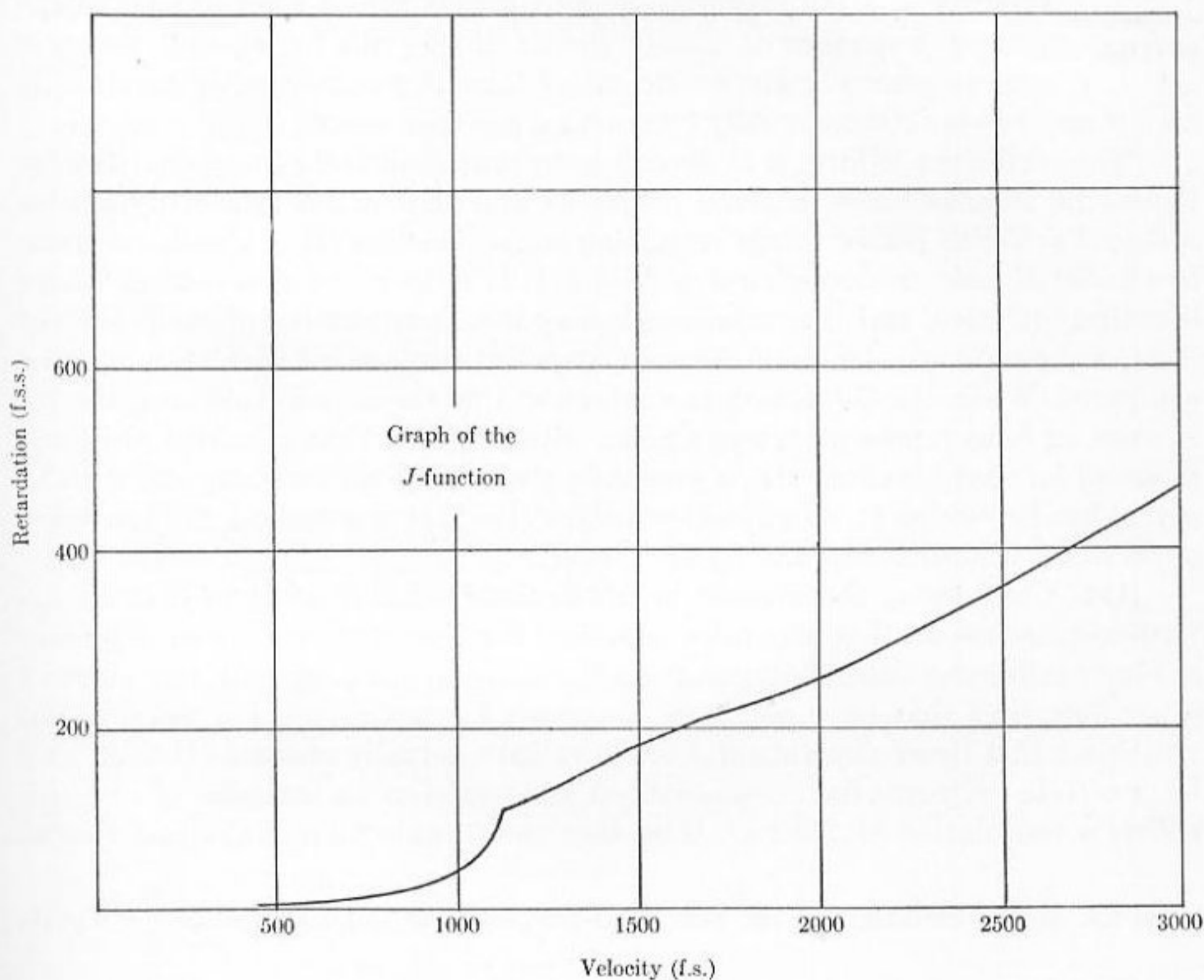


FIGURE 9

The graph of the velocity-retardation function obtained with one of these types is shown in Figure 9 (labeled  $J$ -function, this being the term used by the U. S. Army for the recently determined functions). It is to be noted that the latter shows an abrupt change practically at the velocity of sound, and that the branches of the curve lying on either side of this velocity are very dissimilar. These features are characteristic of all of the recently determined functions.

412. In addition to being a function of velocity, retardation evidently must depend also on certain properties of the projectile and of the resisting medium.

We shall now examine these in turn.

Effect of  
weight  
and diameter  
of projectile

It is a well-known physical law that, other things remaining equal, the retardation suffered by a moving body in a resisting medium is inversely proportional to the body's weight; we may

state, then, that *the projectile's retardation varies inversely as its weight  $w$ .*

It is generally assumed that, other things remaining equal, retardation is directly proportional to the cross-sectional area of the moving body (taken normal to the direction of its motion). Since the area of the projectile's cross-section is directly proportional to the square of the projectile's diameter  $d$ , we may state that the projectile's retardation varies directly as the square of its diameter, or  $d^2$ .\*

413. We know also that the retardation of a projectile depends upon its form, and to account for this we adopt a factor which is called the *coefficient of form* and denoted by the symbol  $i$ . For reasons already outlined above (art. 408) we cannot expect to assign to the coefficient of form a value based entirely on a direct measure of dimensions of the projectile. Certain measurable features of a projectile are found to constitute a basis of comparison among projectiles of closely similar shape, but for reasons that will appear presently the coefficient of form is treated wholly empirically and its measure is obtained wholly from actual performance. †

The coefficient of form

The coefficient of form is so chosen as to express directly the proportion between the retardation of a given projectile and that of an arbitrarily chosen *standard projectile* (other things remaining equal), so that *the projectile's retardation varies directly as its coefficient of form  $i$* . It is to be noted that this coefficient is entirely relative, and that a projectile may have any number of coefficients of form, a different one for each different standard projectile to which it may be compared. When the  $G$ -function is used, as will be the case in this text, the coefficient of form represents a comparison with respect to the standard projectile assumed for that function, viz., a projectile about three calibers long and with an ogival head rounded to a two-caliber radius (the Gâvre standard did not differ appreciably from that assumed by Mayevski).

414. Considering the manner in which the coefficient of form is to be determined, it follows that the value obtained for this coefficient must represent not only influences depending purely on the shape of the projectile, but also any other influences that have not been accounted for separately. For example, let us assume that by an experimental firing we have actually observed that at 2500 f.s. a certain projectile having a weight of one pound and a diameter of one inch suffers a retardation of 362 f.s.s. Also that the  $G$ -table (art. 411) states that at

2500 f.s. the retardation of the standard projectile is 724 f.s.s. Since  $\frac{w}{d^2}$  equals

unity for the projectile fired, it is necessary to assign to it the value  $i = .500$  in order to account for the fact that its observed retardation is only one-half that of the standard. It should be clear, however, that any false assumptions in the

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\* Some authorities disagree with this assumption and cite experimental results which indicate that for projectiles of large cross section the resistance per unit of area is less than the resistance per unit of area for projectiles of small cross section. (Ref. pp. 36-38, *Handbook of Ballistics*, Cranz and Becker). The assumption given above is, however, commonly adopted both here and abroad. Any discrepancies that may result from it are, of course, reflected in the coefficient of form.

† According to the French ballisticians, Hélie, the coefficient of form of an ogival pointed projectile varies in direct proportion to the sine of the semi-ogival angle of the head, i.e., the angle between the projectile's axis and the tangent to the ogive where it intersects this axis. In French methods this relation is used, and the coefficient of form is a function of the semi-ogival angle, viz.,  $\sin \gamma$ . In this country, however, the coefficient of form merely expresses a ratio of comparison between a given projectile and a standard projectile, and this ratio is not derived from any single feature of the projectile but from the actual performance of the projectile as a whole.

entire scheme of arriving at the conception of a coefficient of form, as well as all inaccuracies involved in the determination, will be reflected in the value obtained. The significance of this will become more evident as we proceed to the practical methods of computing trajectories.

415. The artificiality of the coefficient of form is readily seen if we seek to evaluate it, for the same projectile, according to more than one velocity-retardation function. Continuing with the above example we find, according to Mayevski's functions, that at 2500 f.s. the retardation of the standard projectile is 746 f.s.s. Since the observed retardation of the given projectile is actually 362 f.s.s.

at 2500 f.s., we must, in order to secure agreement with Mayevski's functions, assign to this projectile a coefficient of form equal to  $\frac{362}{746}$ , or  $i = .485$ . The difference between  $i = .500$  as obtained according to the  $G$ -function, and  $i = .485$  as obtained according to Mayevski's functions, evident-

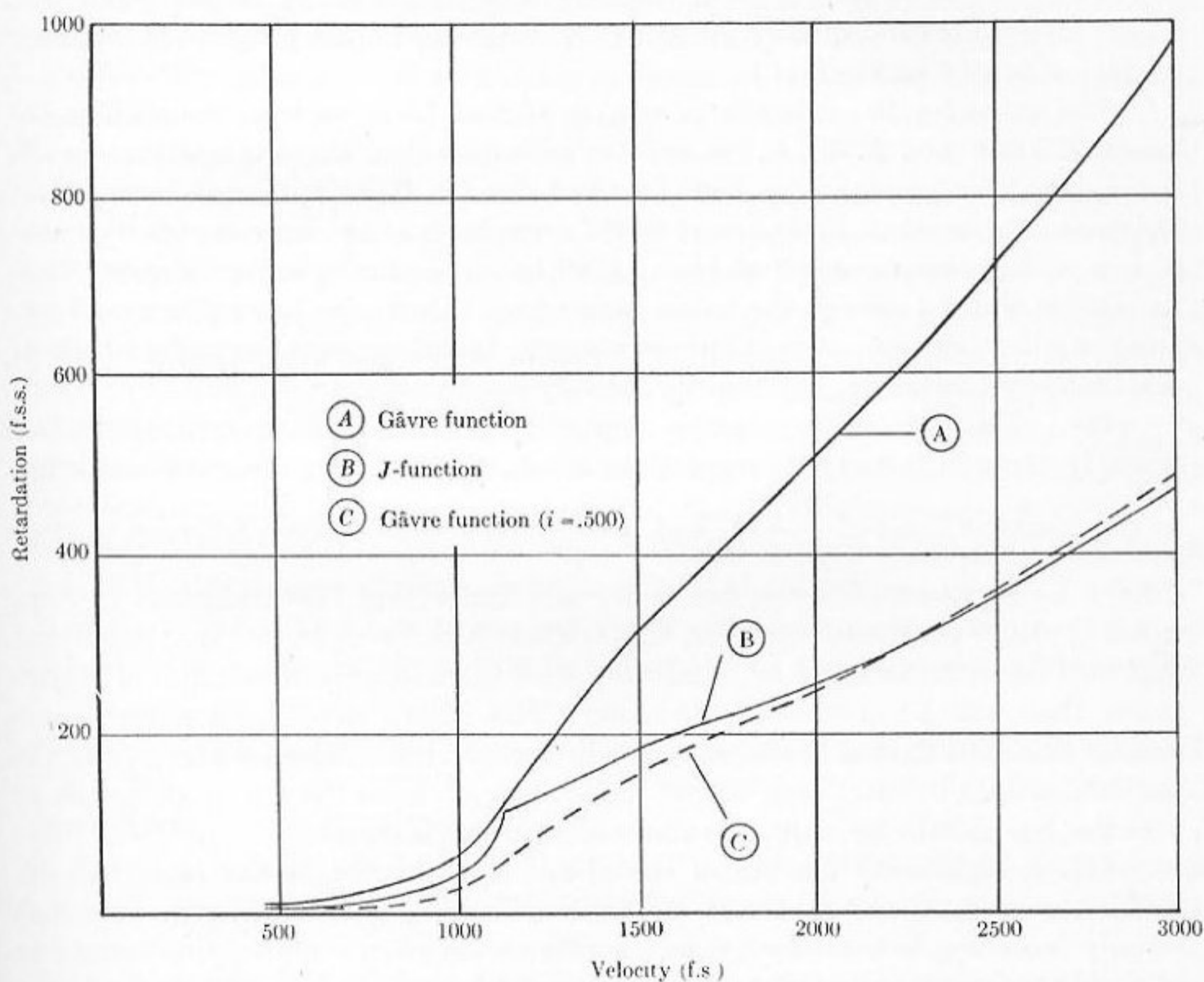


FIGURE 10

ly cannot be due to the projectile itself (since the same projectile at the same velocity has been considered in both cases); it is due entirely to disagreement between the Gâvre and Mayevski functions at 2500 f.s.

In order further to illustrate this important feature of the coefficient of form, let us examine the difference, in terms of coefficient of form, between the  $G$ -function and  $J$ -function; for this purpose we will refer to the graphs of these functions, which already have been shown in Figures 8 and 9, respectively, and in Figure 10 are shown drawn to the same scale. Curve A in Figure 10 is the graph of the  $G$ -function unmodified, i.e., it represents the performance of the

Gâvre standard projectile as defined by the  $G$ -function. Curve  $B$  similarly represents the performance, as defined by the more recent  $J$ -function, of a different standard projectile which we will call the  $J$ -projectile.

The very great discrepancy between the two curves is due first of all to the fact that they represent different projectiles. The  $J$ -projectile is about 5 calibers long, with a conical head about 2 calibers long and tapering almost to a point, and with the rear end "boat tailed" for a length of about one-half caliber from the rear face of the projectile; the resistance suffered by it therefore is materially less than that suffered by the comparatively blunt, square-tailed Gâvre projectile. Let us see now whether by means of a single coefficient of form we can make the Gâvre function account for the actual performance of the  $J$ -projectile. Assuming that the  $J$ -projectile has a coefficient of form  $i = .50$  (as compared to the Gâvre projectile), and on this basis determining its retardation from the Gâvre function, we can plot a graph showing what the retardation of the  $J$ -projectile should be according to the  $G$ -function, and compare it with the graph of the  $J$ -function which represents the actually measured retardations of the projectile. This new graph is labeled  $C$  in Figure 10.

The value  $i = .50$  evidently is exactly correct for a velocity somewhere between 2200 f.s. and 2300 f.s. For greater velocities than this we find that  $i = .50$  is somewhat too great, and at 3000 f.s. the value  $i = .48$  is required to bring about the desired agreement. From about 2200 f.s. to about 1120 f.s. the necessary value of  $i$  increases, becoming .60 at about 1500 f.s., and almost unity at about 1120 f.s. Below about 1120 f.s. the value decreases to about  $i = .75$  at about 900 f.s., and for still lower velocities it increases again, becoming equal to unity at about 500 f.s. and remaining at about unity for very low velocities.

416. The above investigation amounts practically to a comparison between the actually observed retardations of two different projectiles (each reduced to the common basis of  $\frac{w}{d^2} = 1$ ), and it shows that the relation between the two cannot be represented by a constant, and hence that the coefficient of form which we have chosen to represent this relation is not a constant. It is important that this be borne in mind in connection with the methods of solution of trajectories that will be developed presently. The above investigation represents greater extremes than are met ordinarily in our own immediate problem, since the projectiles used by our Navy do not differ as much from the Gâvre projectile as does the  $J$ -projectile on which the above comparison is based.

417. A significant feature of the above investigation is the fact that the coefficient of form, as a factor of comparison between projectiles, may vary materially according to the velocity. In the illustration offered above the variations of  $i$  with velocity are heightened because of the relatively great dissimilarity between the projectiles considered, but variations of like nature, although of less amount, are found even when more nearly similar projectiles are considered. The reason for this is to be found in the existence of numerous influences attending the flight of a projectile, which are not fully expressible in terms of the factors chosen to make up the ballistic coefficient, nor, indeed, in terms of any factors which might be added thereto without greatly complicating matters. Let us consider, for example, the position (or attitude) of a projectile in flight. It is known that the axis of a projectile fired from a rifled gun oscillates about the tangent to the trajectory.\* For a well-designed pro-

Causes of variations in  $i$

\* The motion of the projectile with respect to the tangent will be studied further in Chapter 9.

jectile these oscillations are of small amplitude and the axis remains always close to the tangent, but nevertheless the projectile is always somewhat oblique to the direction of its flight, rather than exactly head-on. This situation invalidates to some degree our assumption that the cross-sectional area opposed to the resisting medium, and the retardation resulting therefrom, are proportional to  $d^2$ . The effective area evidently depends also on the obliquity of the projectile with respect to its direction of flight.

The obliquity of a projectile to its direction of flight at any point in its trajectory depends, in general, upon its stability at that point and hence is influenced by all of the factors affecting stability. These, in addition to constant physical features of the projectile, include its rate of spin and its velocity. For a given stability the obliquity depends also on the rate of change of curvature of the trajectory at the point in question. Since no other factor separately accounts for any of these influences, the coefficient of form evidently must include all of them. The dependence of the coefficient of form on velocity is readily apparent from the above. Also, for the same projectile and same initial velocity we may expect the coefficient of form to vary with angle of departure, since both the curvature and the limits of velocity included within a given trajectory are influenced by the angle of departure.

The effect of obliquity to the direction of flight is not confined to an alteration of the effective cross-sectional area of the projectile. Unequal pressures on the sides of the projectile cause variations in flight. This is well evidenced by the drift, which is the result of lateral components due to obliquity; obliquity may equally well give rise to vertical components, and affect the range in this manner as well as through its influence on the retardation. It is to be appreciated, therefore, that the coefficient of form is influenced by many characteristics of the entire trajectory, and that *its evaluation ultimately must depend upon measurements applied to entire trajectories.*

418. Next to be considered are variations in the resisting medium itself, i.e., in the atmosphere; these may be classified as follows:

- |                                    |  |
|------------------------------------|--|
| Variations<br>in the<br>atmosphere | <ul style="list-style-type: none"> <li>(a) Variations in the density of the atmosphere near the surface of the earth, as determined by the air temperature and barometric pressure near the surface.</li> <li>(b) Variations in the density of the atmosphere with altitude, due to the variations in temperature and pressure which are incident to changes in altitude.</li> <li>(c) Variations in the density of the atmosphere due to variations in its saturation with moisture (i.e., in its relative humidity), which depend both on surface conditions and on altitude.</li> <li>(d) Variations in the elasticity of the atmosphere, due to variations in its temperature alone, which depend both on surface conditions and on altitude.</li> </ul> |
|------------------------------------|--|

It will be noted that the variations enumerated under (a), (b), and (c) above are of the same kind, i.e., they are all variations in the *density* of the atmosphere, although they result from different causes. A difference in the cause of a given density variation occasions no difference in its effect upon retardation; that is to say, a given percentage variation in air density has a like effect on retardation whether the variation be due to one or another of the above causes. The above distinction is made, however, in view of differences in procedure of taking into account the density variations due to the different causes enumerated. On the other hand, it is to be noted that the elasticity effect mentioned under (d), which depends upon the air temperature just as do the density effects (a), (b), and (c),

is nevertheless entirely independent of the latter; i.e., the air temperature affects density and elasticity both simultaneously but each independently. The effects on the retardation due to these several variations in the atmosphere, and the methods by which they are taken into account, will now be examined in turn.

419. It is generally supposed that the retardation of a projectile varies directly in proportion to the density of the air; this theory never has been substantiated by experimental proof, but it is, nevertheless, commonly accepted by ballisticians. Under this assumption it is possible to express the ratio between retardations under different conditions of air density, by means of a factor which expresses the ratio between the densities themselves. In other words, a given percentage variation in air density is assumed to cause a like percentage variation in retardation.

For tabulated retardation functions, a standard surface air density is assumed. Practice varies somewhat as to the choice of this standard density, but

for tables used in the United States the standard is defined by the density of air at a temperature of 15°C. (59°F.), at a barometric pressure of 750 mm. (approximately 29.53 in.) of mercury, and 78 per cent saturated with moisture. The actual weight of air under these conditions is

1.2034 kilograms per cubic meter (approximately .075 pounds per cubic foot).\*

We will denote by the symbol  $\delta_s$  this standard surface density, by  $\delta_a$  the actual surface density under any given set of conditions, and by  $\delta$  the ratio of actual surface density under any given set of conditions to the standard surface density, i.e.,

$$\delta = \frac{\delta_a}{\delta_s}. \quad (404)$$

This surface density factor,  $\delta$ , therefore states the percentage of the standard surface density that is represented by the actual surface density under any given set of conditions.

It follows that if  $\delta > 1$ , the actual density is greater than the assumed standard, and the retardation under the given conditions is greater than under standard conditions, i.e., greater than the tabulated value. We may state, then, that *retardation varies directly as  $\delta$* .†

Values of the ratio  $\delta$  are tabulated in Table III,† *Range and Ballistic Tables, 1935*, against the arguments temperature (F.°) and barometer (inches of mercury),

and for a constant saturation of 78%. Examination of this table will show that the value  $\delta = 1$  corresponds to a great variety of combinations of temperature and pressure, as well as to the combination 59° and 29.53 inches.

This is due to the fact that the density of air varies independently according to both temperature and pressure; hence the effect on density of a change in temperature may be balanced exactly by an accompanying change in pressure. On the other hand, variations of considerable magnitude may result from combinations of low temperature and high pressure, or vice versa, and may have a marked effect on the trajectory.

\* A discussion of the various standards that are, and have been, assumed both here and abroad, is given on pp. 15-18, *T.S. No. 148*, April, 1921, (U. S. Army).

† In the 1926 and 1930 editions of this book, the density factor  $\delta$  was chosen to represent the ratio  $\delta_s/\delta_a$ , i.e., the reciprocal of the above, in conformity with tables which were used in connection with those editions. The present practice, both in the U. S. Army and in the U. S. Navy, is to use tables which give the ratio of actual density to standard density, or the value of  $\delta$  as defined above. The only difference occasioned by this change of procedure is that  $\delta$  now appears in the denominator of the ballistic coefficient (406), instead of in the numerator as heretofore.

420. The density of the atmosphere at a level above the earth's surface depends both upon the density at the surface and upon the altitude of the given level, since both temperature and pressure decrease with altitude. However, it is not a simple matter to define the density for a level aloft in terms of surface density and altitude, since it is known that fluctuations in the surface air density often are caused by local influences which do not extend very far aloft. Similarly, aloft currents may cause changes at aloft levels that are not felt at the surface. It is generally true, however, that density fluctuations decrease with altitude; in other words, that a given percentage variation of surface density with respect to the surface standard is greater than the accompanying percentage variations of aloft densities with respect to the standards assumed for aloft levels.

Accurate determination of aloft densities requires actual measurement of temperature and barometric pressure aloft. However, the general scheme followed in ballistics is to assume a standard altitude-density relation for the computation of basic tables, and to make adjustments for such variations from the assumed standard relation as may be observed in connection with service problems. The standard altitude-density relation used in ballistics is based on the standard surface density  $\delta_s$  previously defined, and is assumed to have the form

$$H_y = 10^{-.00001372y} \quad (405)$$

or

$$\log H_y = - .00001372y$$

in which  $y$  represents the altitude in feet and  $H_y$  the ratio of air density at any altitude  $y$  to air density at the surface. It follows from the above that *retardation varies directly as  $H_y$* . The values of  $\log H_y$  are tabulated in Table II, *Range and Ballistic Tables, 1935*, against  $y$  as argument. This table is often referred to as the *H-table*, and  $H_y$  as the *H-function*.

It may be considered that  $\delta_s$  and  $H_y$  together define a sort of standard atmosphere on which ballistic tables are based.\* It is important to remember that this standard atmosphere implies not only the standard altitude-density relation  $H_y$ , but also the standard surface density  $\delta_s$ ; in other words, that  $H_y$ , as defined by (405), is not applicable directly to other than standard surface density. This follows naturally from the conclusion already stated, viz., that density variations at the surface generally are greater than their accompanying variations aloft. This situation evidently implies that the entire altitude-density relation varies according to the surface density. It is possible, however, to make an adjustment in the observed non-standard surface density itself, so that this adjusted density, when used with the standard altitude-density relation  $H_y$ , may produce the equivalent of the observed density used with an altitude-density relation especially applica-

\* This varies somewhat from the standard atmosphere as accepted by the Bureau of Standards, Weather Bureau, and some other activities, (for which see Table 695, *Smithsonian Physical Tables*, 8th Edition). The form assumed for  $H_y$  is particularly advantageous in connection with certain operations that are applied in the preparation and use of ballistic tables, as will be noted in article 630. The altitude-density relation defined by  $H_y$ , as given above, is applicable to the limits of altitude ordinarily reached by trajectories, but not beyond the limits of the troposphere (about 40,000 feet). For a more complete discussion of the altitude-density relation in connection with ballistics; including its application to levels within the stratosphere, see pp. 172-176, *Introduction to Ballistics* (U. S. Army, 1921), also pp. XVI-XVIII, *Exterior Ballistic Tables Based on Numerical Integration*, Volume I (U. S. Army, 1924)

ble to the latter. This course is much the simpler; among other things, it permits the adjustment both for non-standard surface density and for the non-standard altitude-density relation incident thereto, to be made by means of a change in the surface density factor  $\delta$  alone, rather than by changes both in the latter and in  $H_v$ . The manner in which this is done requires a brief digression on methods of solution of the trajectory which are to be taken up in detail in Chapters 6 and 8.

421. The modern method of computing a trajectory is a step-by-step process which determines points in the trajectory in succession from the origin. The computation for each point is based on the correct physical data applicable to that point, including the air density. Since computations for tables are always based on standard conditions, this means that the density at each point is determined on the basis of standard surface density ( $\delta=1$ ) and of the value of  $H_v$  for the ordinate of the given point, that is, on  $\delta H_v$ , with  $\delta$  equal to unity. Now if a trajectory were to be computed for other than standard conditions by a similar process, we would use at each point the non-standard  $\delta$  multiplied by an appropriate, non-standard  $H_v$ . We can, however, just as well consider  $H_v$  to remain at standard and the necessary change in the product  $\delta H_v$  to be absorbed by  $\delta$  alone. This results in an artificial density factor which is, in fact, a percentage of the standard combination of  $\delta H_v$ , and which varies from point to point. Let us suppose now that a mean is taken of these varying density factors, and substituted in place of the actual surface density factor assumed for our computation. This mean value of  $\delta$  is then a purely fictitious density factor, artificially set up to produce the equivalent, over the trajectory as a whole, of the use of the actual non-standard surface density and of the actual non-standard altitude-density relation applicable to the latter. The density factor determined according to the principles just outlined, is called the *ballistic density factor*, and we shall denote it by the symbol  $\delta_b$ ; it is commonly referred to simply as the *ballistic density*.\*

422. Since direct application of the above process obviously is impracticable for service problems, the following steps have been taken to reduce the process to a practical basis for the determination of ballistic density. For a wide variety of trajectories, the effects of given variations in density considered to apply only within restricted limits of the trajectory, have been compared to the effects of like variations considered to apply to the entire trajectory. From these comparisons *air-density weighting factors* have been deduced for portions of the trajectory lying within various zones of altitude, so that the effects of the several density factors which pertain to the several zones of the trajectory may be duly weighted and combined to give a weighted mean density factor, or *ballistic density*, for the entire trajectory. For example, the air-density weighting factors for a trajectory having a maximum ordinate of 3000 feet are .20 for the zone 0-600 feet, .28 for the zone 600-1500 feet, and .52 for the zone 1500-3000 feet. Now if  $\delta_1$ ,  $\delta_2$ , and  $\delta_3$ , respectively, are the density factors for these zones, then  $.20\delta_1 + .28\delta_2 + .52\delta_3$  is the ballistic density for the trajectory. It is to be understood that the ballistic density accounts for variations from the assumed standard surface density ( $\delta_s$ ) and from the assumed

\* More properly, the ballistic density is the actual density (i.e., weight per unit volume) to which the ballistic density factor corresponds. However, since it is always the density factor, and not the density itself, that is used in practical applications, it is common practice to use the terms "density," "surface density," "ballistic density," etc. when referring, in fact, to the corresponding factors.

standard altitude-density relation ( $H_v$ ); in other words, it states the percentage variation of the actual atmosphere in a given case from the standard atmosphere, within the limits of the given trajectory.

423. The ballistic density for a given case may be found by either of two methods, viz.,

- (a) By actually *measuring* the densities in various zones aloft, and applying the air-density weighting factors to the ratios of these observed densities to the standard densities for their respective zones.
- (b) By *assuming* densities for various zones aloft, on the basis only of the measured surface density, and using these, as in (a) above, in place of values actually observed aloft.

Determination  
of ballistic  
density from  
aloft obser-  
vations

The first of the above methods is the only one which accounts for conditions as they actually exist, and which allows for unpredictable variations in the atmosphere. It is used when aloft observations are available.\* Such observations can be obtained only by specially equipped airplanes, and hence often are not

available.

The second of the above methods is designed to provide for an approximate determination of the ballistic density when facilities for obtaining aloft observations are not at hand, or when conditions are such as to render aerological flights

Determination  
of ballistic  
density from  
surface  
observations

impracticable. It depends upon an assumption of aloft densities under various conditions of surface density, based on a study of a great many observations taken in the past.† In this method only the surface density is determined by actual observation, and the aloft densities corresponding thereto are assumed to follow the average

relation based on past experience, as noted above. In order to simplify matters, tables have been prepared which embody these average relations, and from which the ballistic density corresponding to any surface density and maximum ordinate may be found directly. Extracts from these tables are given in Table IV, *Range and Ballistic Tables, 1935*.

424. The ballistic density for a given trajectory is found from Table IV by entering the latter with the surface density factor as horizontal argument and the maximum ordinate of the trajectory as vertical argument. This

requires that the surface density factor, i.e.,  $\delta$ , first be found from Table III, with the surface temperature and pressure as arguments. An illustration will make this clear.

*Given:* Surface temperature, 84°F.; surface barometric pressure, 29.90 inches.

*Find:* The ballistic density for a trajectory having a maximum ordinate of 18,000 feet.

From Table III, with the arguments 84° and 29.90 inches, we find

$$\delta = .960.$$

From Table IV, with the arguments  $\delta = .960$  and  $y_s = 18,000$  feet, we find

$$\delta_b = .993.$$

\* At the present time the observed ballistic density for various maximum ordinates is determined by aerological units attached to force flagships, and transmitted to vessels in the vicinity of the latter. A complete description of the details involved in this determination is given in *Technical Regulations No. 1236-1* (U. S. War Department, 1934).

† Ref. *Aeronautical Meteorology*, Gregg; also *Monthly Weather Review, Supplement No. 20* (U. S. Weather Bureau publication no. 768, May, 1922).

This example shows that although in this case the variation from standard density is 4.0% at the surface, the weighted mean variation for the entire trajectory is only 0.7%, or about one-sixth of that indicated at the surface. Inspection of other values in the table indicates that the range of variations in aloft densities is materially less than that in the corresponding surface densities, which is in accord with the conclusions previously stated. For example, for values of  $\delta$  which vary from .900 to 1.100, the corresponding values of  $\delta_b$ , for a trajectory having a maximum ordinate of 18,000 feet, vary only from .967 to 1.054.\*

425. It seems appropriate at this point to make a general appraisal of what may reasonably be expected, in the way of accuracy, from the whole scheme for taking into account variations in air density, as outlined in the foregoing articles. First of all, even when aloft densities actually are measured, they are combined into a weighted mean by the application of weighting factors which are designed to serve for a wide variety of trajectories and which must, therefore, constitute a rather broad average. Considering, however, the degree of approximation that must be accepted in the basic density observations themselves, under practical service conditions, the use of anything more elaborate than the average weighting factors hardly is justifiable. Table IV involves not only the average weighting factors, but also a broad average of observations designed to apply without regard to time of day, locality, climate, season, or numerous other factors that may influence the altitude-density relation. Table IV therefore can be regarded only as a substitute for the better determinations which are based on actual observations.

It may be accepted that any errors incident to the approximations entering into the determination of a ballistic density from *actual aloft observations*, as outlined above, will be of a small order in comparison with other errors of gunfire. The order of error involved in the use of Table IV, although it may materially exceed that of the observational method, nevertheless has every likelihood of being much smaller than that incident to the use only of the surface density factor.†

426. Atmospheric density varies only slightly with humidity, and the effect of such variations from the standard humidity as are likely to occur in practice may be disregarded altogether. Variations in atmospheric density due to variations in humidity, within the probable extremes of the latter over sea areas, are limited to about ( $\pm$ ) 0.3%,‡ with a corresponding, limiting effect of only about 0.1% on the range of a trajectory.

427. The temperature of the air, entirely apart from its effect on atmospheric density, affects the elasticity of the air. The latter, in turn, affects the

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\* Under the assumptions made in arriving at the conception of a ballistic density, the latter, as determined from these tables, would be expected to be equal to unity at all altitudes when the surface density is at standard (i.e.,  $\delta = 1$ ). That it does not actually fulfill these conditions is due to the fact that the standard altitude-density relation ( $H_v$ ) assumed to apply to standard surface density ( $\delta = 1$ ), does not agree exactly with the corresponding observed relation which is embodied in Table IV. The slight variation of  $H_v$  as assumed for ballistic tables, from the observed average altitude-density relation for  $\delta = 1$ , can be accounted for by employing the ballistic density corresponding to  $\delta = 1$  in connection with values from the ballistic tables.

† See also art. 1013.

‡ A table showing the effect on atmospheric density of variations in humidity, is given in Table VIII, *Technical Regulations No. 1236-1* (U. S. War Department, 1934).

The temperature-elasticity effect

velocity of wave propagation in air (i.e., the velocity of sound). The dependence of the retardation function on the velocity of sound already has been noted in articles 409 and 411. A change in the velocity of sound causes a change in the retardation function throughout its entire length, but the effect of this change is much greater in the critical portion in the vicinity of the velocity of sound than elsewhere. This may be seen by examining the graph of the  $J$ -function (Figure 9). There is every reason to believe that the abrupt change in this curve is closely associated with the velocity of sound, and hence that the point where this abrupt change occurs will shift as the velocity of sound shifts. At the steep portion of the curve just preceding the velocity of sound, even a small shift has an appreciable effect on the relation between retardation and velocity, although elsewhere the effect of a similar shift is much less.

It will be apparent that a correction for the effect of a variation in elasticity due to a change in temperature, must be treated as a correction to the retardation function itself, and hence it takes on all the complications of the latter. Fortunately the effect of variations in elasticity, within the limits of such variations and of the velocities normally encountered in practice, is small, and no serious error results from neglecting it.\* For high-velocity guns the amount of this effect on the entire trajectory, measured in terms of the percentage variation in total range resulting therefrom, does not exceed about .02% per degree (F.) of variation from the standard temperature (59°). Through its operation on the air density, the effect of the same one degree of variation in temperature may be as great as about seven times the above.

428. For convenience in notation we will now combine the surface density factor and the several factors pertaining to the projectile as follows.

The ballistic coefficient  $C$

$$C = \frac{w}{\delta id^2} \quad (406)$$

$C$  is called the *ballistic coefficient*, and it is a measure of comparison between the retardation of a given projectile in air of a given surface density, and the retardation of the standard projectile in air of standard surface density. According to the relation stated above, *retardation varies inversely as the ballistic coefficient,  $C$ .*

We may now consider the values of  $G_v$  in the  $G$ -table to express retardation for the conditions under which  $C = 1$ , and for any given set of conditions the retardation, according to the  $G$ -function, is defined by

$$R_\alpha = \frac{G_v}{C} \quad (407)$$

For example, to find the retardation at 3000 f.s. for a 5" projectile whose weight is 50 lbs. and coefficient of form .600, in air for which  $\delta = .900$ , we have

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\* Consideration of the temperature-elasticity effect has entered the field of exterior ballistics only comparatively recently, and subsequently to the determination of the  $G$ -function. The latter therefore takes no account of this effect and retardations tabulated in the  $G$ -table necessarily represent an average elasticity corresponding to the range of temperatures under which the Gâvre firings were conducted. As the normal average of temperatures encountered in practice probably varies but little from the Gâvre average, any error resulting from omission of a temperature-elasticity correction for variation of surface temperature from the assumed standard (59°F.) should remain small. For trajectories which ascend to great altitudes and hence encounter materially lower temperatures aloft, such a correction becomes of greater significance, and the most recent ballistic tables include it.

$$C = \frac{50}{.900 \times .600 \times 25} = 3.7037.$$

In Table I, at 3000 f.s., we find  $G_v = 973.0$  f.s.s., whence the retardation for the assumed case is

$$R_\alpha = \frac{973.0}{3.7037} = 262.71 \text{ f.s.s.}$$

429. Range tables are always based on standard air density at the surface (i.e.,  $\delta = 1$ ) and on the standard altitude-density relation defined by  $H_v$ . The special symbol  $E$  is used to define retardation under these conditions; that is,  $E$  denotes the retardation of a given projectile due to the resistance of air of standard density at the surface but corrected for variation of density with altitude, whence

$$E = \frac{G_v \times H_v}{C} \quad (408)$$

in which  $C$  is always based on  $\delta = 1$ . For example, in making a range-table computation for a 5" gun whose projectile weighs 50 lbs. and has a coefficient of form equal to .600, the retardation at a point in the trajectory where the remaining velocity is 2000 f.s. ( $v = 2000$ ) and the altitude is 1000 feet ( $y = 1000$ ), is found as follows. From Table I we find  $G_v = 510.5$  f.s.s. From Table II we find  $\log H_v = 9.98628 - 10$ . With  $\delta = 1$  we compute  $C = 3.3333$ .

Determination of retardation for range-table computations		
$G_v = 510.5$	.....	$\log 2.70800$
$H_v =$	.....	$\log 9.98628 - 10$
$C = 3.3333$	.....	$\text{colog } 9.47712 - 10$
$E = 148.39$ f.s.s.	.....	$\log 2.17140$

430. The following is an elementary example of the process of finding the coefficient of form.

*Given:* A 6" projectile weighing 105 lbs. was fired practically horizontally through two pairs of screens. The distance from the gun to the midpoint of the first pair was 400 feet, and to the midpoint of the second pair 915 feet. The mean of ten shots gave as the mean velocity between the first pair 2211.7 f.s. and between the second pair 2182.3 f.s. The temperature of the air was 57°F. and the barometer 29.75 inches.

*Find:* The coefficient of form of this projectile according to the  $G$ -function.

We may proceed to determine the retardation from the experimental data exactly as was done in article 404.

$(v_1 - v_2) = 29.4$	.....	$\log 1.46835$
$(v_1 + v_2) = 4394$	.....	$\log 3.64286$
$2l = 1030$	.....	$\log 3.01284$
$R_\alpha = 125.42$ f.s.s.	.....	$\text{colog } 6.98716 - 10$
		$\log 2.09837$

The actual retardation was therefore 125.42 f.s.s. at the velocity 2197 f.s. (approximately). According to the  $G$ -function the retardation of the standard projectile at 2197 f.s. is  $G_v = 591.7$  f.s.s. The disagreement between the experimentally measured value and the tabular value must be accounted for by the ballistic co-

efficient, and since in the latter we can directly evaluate  $\delta$ ,  $w$ , and  $d^2$ , any remaining disagreement must be assigned to  $i$ . We will therefore expand (407) and then rewrite it in terms of  $i$ , as follows

$$R_\alpha = G_v \times \frac{\delta i d^2}{w}$$

$$i = \frac{R_\alpha}{G_v} \times \frac{w}{\delta d^2} \tag{409}$$

We may now solve (409) to find the value of  $i$  that is required to establish the required agreement between the measured retardation and the  $G$ -function. From Table III we find  $\delta = 1.012$ .

$R_\alpha = 125.42$		log 2.09837
$\delta = 1.012$	log 0.00518	colog 9.99482-10
$w = 105$		log 2.02119
$G_v = 591.7$	log 2.77210	colog 7.22790-10
$d^2 = 36$	log 1.55630	colog 8.44370-10
<u><math>i = .61092</math></u>		log 9.78598-10

The coefficient of form we have thus found, practically .61, has only a limited application. For other trajectories than the one here involved it may differ somewhat, for reasons that have already been gone into at length (arts. 413-416). But even a very elementary determination of this type may be useful for a first approximation of the coefficient of form of a projectile.

EXERCISES

1. Compute the values of  $\log C$  for the cases listed in the following table.

Problem	DATA					ANSWERS
	Projectile			Atmosphere (surface)		log $C$
	$d$ (in.)	$w$ (lbs.)	$i$	Temp. (°F.)	Bar. (in.)	
1.....	3	13	1.00	61	29.80	0.15710
2.....	4	33	0.67	65	29.60	0.49268
3.....	5	50	0.59	57	30.25	0.51734
4.....	6	105	0.61	70	30.50	0.67524
5.....	8	260	0.61	85	29.75	0.84437
6.....	12	870	0.61	93	30.20	1.01811
7.....	14	1400	0.70	69	29.80	1.01401
8.....	16	2100	0.61	32	30.15	1.09482

2. Find the surface density factor  $\delta$  (Table III), and the ballistic density  $\delta_b$  (Table IV), for the following cases.

Problem	DATA			ANSWERS	
	Atmosphere (surface)		Maximum ordinate (feet)	$\delta$	$\delta_b$
	Temp. (°F.)	Bar. (in.)			
1.....	65	29.60	1000	.990	.991
2.....	85	29.75	18000	.953	.991
3.....	57	30.25	8000	1.030	1.026
4.....	69	29.80	13000	.988	1.003
5.....	32	30.15	15000	1.081	1.050

3. Given the measured velocities of a projectile at two points, as determined by firing horizontally through screens, compute the retardation and resistance due to the atmosphere, and state the velocity (approximately) to which the results apply ( $g = 32.16$ ).

Problem	DATA				ANSWERS		
	$w$ (lbs.)	Distance between points of measure- ment (ft.)	Measured velocities at		$R_a$ (f.s.s.)	$R_f$ (lbs.)	Apply to velocity (f.s.)
			First point (f.s.)	Second point (f.s.)			
1.....	13	500	2680	2572	567.20	229.28	2626
2.....	50	500	3140	3088	323.85	503.50	3114
3.....	870	500	2870	2854	91.58	2477.5	2862
4.....	2100	500	2590	2578	62.02	4049.4	2584

4. Find the retardation of a projectile at the earth's surface, according to the  $G$ -function, for the following cases.

Problem	DATA					Velocity (f.s.)	$R_a$ (f.s.s.)
	Projectile			Atmosphere (surface)			
	$d$ (in.)	$w$ (lbs.)	$i$	Temp. (°F.)	Bar. (in.)		
1.....	4	33	0.67	65	29.60	2890	293.82
2.....	6	105	0.61	70	30.50	2584	160.92
3.....	8	260	0.61	85	29.75	2730	119.06
4.....	14	1400	0.70	69	29.80	2580	73.58
5.....	16	2100	0.61	32	30.15	2565	60.52

5. Given the measured velocities of a projectile at two points, as determined by firing horizontally through screens, compute the coefficient of form according to the  $G$ -function. (Use  $V$  to nearest f.s. in entering  $G$ -table.)

Problem	DATA							ANSWERS
	Projectile		Atmosphere (surface)		Measured velocities at		Distance between points of measure- ment (ft.)	
	$d$ (in.)	$w$ (lbs.)	Temp. (°F)	Bar. (in.)	First point (f.s.)	Second point (f.s.)		
1.....	3	13	85	29.75	2660	2562	500	1.0015
2.....	5	50	32	30.15	3120	3066	500	.60272
3.....	12	870	57	30.25	2840	2824	500	.60187
4.....	16	2100	69	29.80	2610	2599	500	.61687

## THE DIFFERENTIAL EQUATIONS OF MOTION OF A PROJECTILE IN AIR AND SIACCI'S METHOD OF SOLVING THEM.

### New Symbols Introduced

- $\rho$  . . . . . Radius of curvature of the trajectory at any point.
- $ds$  . . . . . Differential of the length of arc of the trajectory.
- $f(v)$  . . . . . A general notation for the function of velocity taken to express the retardation of a projectile due to air resistance, under the condition  $C = 1$ .
- $u$  . . . . . The pseudo velocity; a component of the remaining velocity obtained by projecting the latter vertically upon the line of departure or a line parallel thereto.
- $f_a$  . . . . . The altitude factor in Siacci's Method; the ratio of air density at the surface to air density at the height of the mean ordinate of the trajectory.
- $\beta$  . . . . . An approximate constant devised by Siacci to reduce the differential equations to an integrable form.
- $C_a$  . . . . . The Siacci  $C$ ; the ballistic coefficient augmented by Siacci's approximate mean-value constants  $f_a$  and  $\beta$ , and used only with Siacci's Method.
- $S_u, T_u, I_u, A_u$  . . . . These represent integral expressions occurring in Siacci's Method, which have been tabulated.

501. The motion of a projectile in air, imparted to it initially by the gun, is modified in flight by two forces, namely, the force of gravity and the force of air resistance. In terms of their corresponding accelerations, and according to the

general law  $F = \frac{w}{g} \alpha$ , these forces are represented by the expressions,  $\frac{w}{g} \times g$ ,

or  $w$ , for the force of gravity, and  $\frac{w}{g} \times R_\alpha$  for the force of air resistance. The

trajectory being curved, as a consequence of the action of gravity, a centrifugal force is set up; the latter, for any point in the trajectory where the radius of curvature is  $\rho$  and the remaining velocity  $v$ , is represented by the expression  $\frac{w}{g} \times \frac{v^2}{\rho}$ . The force of gravity operates vertically and the force of air resistance in the direction of the tangent to the trajectory, and the direction of the centrifugal force is normal to the latter.

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\* Ref. any standard work on mechanics.

At any point  $P$  in the trajectory (Figure 11) the force of air resistance has a zero component in the direction normal to the tangent at that point, i.e., along the radius of curvature  $OP$  at that point. The angle of inclination at the given point being  $\theta$ , the force of gravity has in this same direction the component  $w \cos \theta$  acting directly toward the center of curvature, and the latter is exactly balanced by the centrifugal force acting directly away from the center of curvature. We have, then,

$$\frac{w}{g} \times \frac{v^2}{\rho} = w \cos \theta$$

$$\frac{v^2}{\rho} = g \cos \theta. \quad (501)$$

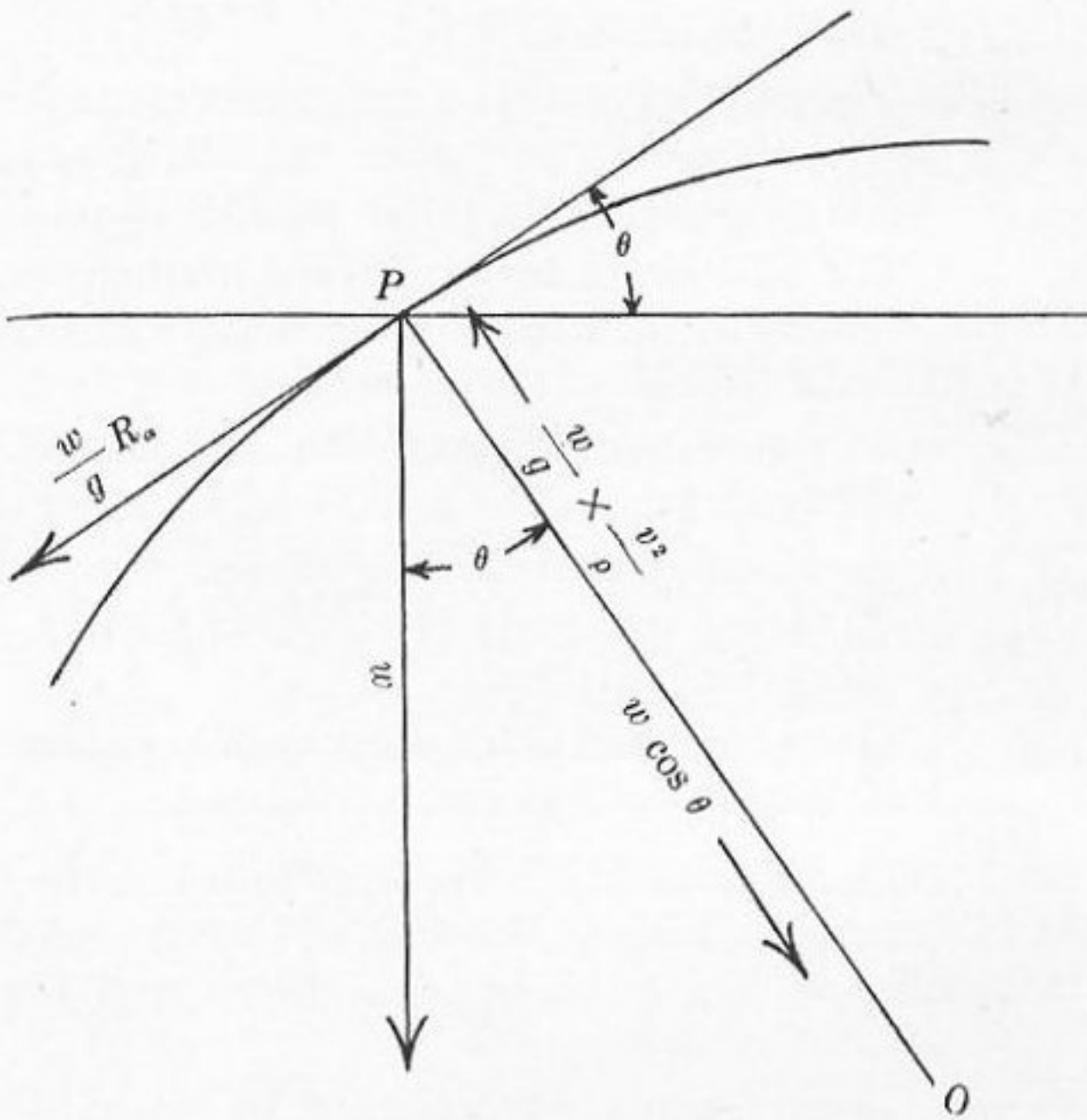


FIGURE 11

In terms of  $s$ , denoting length measured along the arc of the trajectory, the radius of curvature of the trajectory at point  $P$  is  $\rho = -\frac{ds^*}{d\theta}$ ; substituting this value in

(501), and transposing, we have

$$ds \cos \theta = -\frac{v^2}{g} d\theta.$$

\* Ref. any standard work on mechanics.

Since  $ds \cos \theta = dx$ , we may write from the above

$$dx = -\frac{v^2}{g} d\theta \quad (502)$$

Also, since  $dy = dx \tan \theta$ , we may write from (502)

$$dy = -\frac{v^2}{g} \tan \theta d\theta. \quad (503)$$

Dividing both sides of (502) by  $dt$

$$\frac{dx}{dt} = -\frac{v^2}{g} \frac{d\theta}{dt}$$

and since  $\frac{dx}{dt} = v \cos \theta$ , we have

$$v \cos \theta = -\frac{v^2}{g} \frac{d\theta}{dt}$$

whence

$$dt = -\frac{v}{g} \sec \theta d\theta. \quad (504)$$

**502.** We have already found that the retardation of a projectile due to air resistance is not expressible in any simple form, but we know that it is a function of the velocity. Mayevski's functions state this in a formal manner by means of the expression (403), while the Gâvre function states it in tabular fashion. For convenience, we shall use the notation  $f(v)$  (i.e., a function of the velocity) to express the retardation due to air resistance for the condition  $C = 1$ . We also have found that the retardation due to air resistance is inversely proportional to the ballistic coefficient, which combines the various constants that pertain to the projectile and atmospheric conditions of any particular case. Acceleration is denoted by  $\frac{dv}{dt}$ , and we may therefore write

$$\frac{dv}{dt} = -\frac{1}{C} f(v).$$

Also, since  $v \cos \theta$  denotes the horizontal velocity we may write for the horizontal component of acceleration

$$\frac{d(v \cos \theta)}{dt} = -\frac{1}{C} f(v) \cos \theta.$$

Substituting for  $dt$  in the above its value as found from (504), we have

$$\frac{g d(v \cos \theta)}{-v \sec \theta d\theta} = -\frac{1}{C} f(v) \cos \theta$$

or

$$d(v \cos \theta) = \frac{1}{C} \frac{v}{g} f(v) d\theta. \quad (505)$$

503. We now have derived the following differential equations, which state the relations among several elements of the trajectory in air.

$$dx = - \frac{v^2}{g} d\theta \quad (502)$$

Differential equations pertaining to the trajectory in air

$$dy = - \frac{v^2}{g} \tan \theta d\theta \quad (503)$$

$$dt = - \frac{v}{g} \sec \theta d\theta \quad (504)$$

$$d(v \cos \theta) = \frac{1}{C} \frac{v}{g} f(v) d\theta. \quad (505)$$

Each of the equations (502), (503), and (504) contains the variables  $v$  and  $\theta$ , in addition to the variables to be solved for (i.e.,  $dx$ ,  $dy$ ,  $dt$ ). Equation (505) gives a relation between  $v$  and  $\theta$ , and the solution of the entire system represented by the four equations depends upon the solution of (505). We shall therefore refer to (505) as the *chief equation*;<sup>\*</sup> for if values of  $v$  corresponding to any given values of  $\theta$

are available, then (502), (503), and (504) can be integrated within any limits of  $\theta$ , and the values of  $x$ ,  $y$ , and  $t$  for any point in the trajectory may be determined. The solution of the chief equation, however, depends in turn upon knowledge of the retardation function, represented by  $f(v)$ , which appears in it. We have already seen that  $f(v)$  itself is a very complex function, and hence many difficulties are involved in the solution of the chief equation.

504. Historically, the methods of solution of the chief equation went through stages closely related to those we have already noted in the development of the retardation function (art. 409). The earliest methods concerned themselves with solutions under monomial laws of retardation, that is, laws assuming retardation to vary as the square, cube, fourth power, etc., of the velocity. As early as 1719, Bernoulli stated the integrations in terms of a law assuming retardation to vary as the  $n$ th power of velocity, that is, for  $f(v) = kv^n$  ( $n$  being an integer). In 1744, d'Alembert gave a solution for a more general law of the form  $f(v) = kv^n + q$ . A further mode of attack, designed to establish laws which would more accurately represent the known velocity-retardation relation and yet leave it possible to apply formal integration to the chief equation, lay in deriving from the latter numerous forms which would serve the purposes of integration, and in attempting to fit the velocity-retardation relation into one of them. Numerous such forms were offered from time to time, Siacci alone having given out fourteen of them, but no advantages were derived from this mode of attack. The significant feature of all of these early methods was the attempt to reduce the retardation function to a form permitting formal integration of the chief equation in a single

Exact solution of the chief equation in terms of an approximate retardation law

\* If values of  $v$  and their corresponding values of  $\theta$  be laid off as vectors from a common origin, with  $v$  as the vector length and  $\theta$  as its inclination with respect to the axis of coordinates, the curve connecting the extremities of these vectors is called, in mechanics, the *hodograph* of the motion thus represented. Equation (505) is the equation of the hodograph for the motion of a projectile in air.

step; that is, these methods all resorted to approximate forms of the retardation function, although adhering otherwise to exact solutions of the chief equation.

505. The general acceptance of the system of establishing zones of velocity, with a separate monomial law for each zone, was followed, naturally, by methods of solution appropriate to this scheme. As has already been noted (art. 409), the most generally favored of the systems based on zones was Mayevski's series,

which is represented by the general expression  $R_\alpha = \frac{A}{C} v^a$ , with  $A$  and  $a$  varying

from zone to zone but remaining constant within a given zone. To apply this expression to the chief equation we replace  $f(v)$  with  $Av^a$  and have

The chief equation  
in terms of  
Mayevski's functions

$$d(v \cos \theta) = \frac{1}{C} \frac{A}{g} v^{(a+1)} d\theta. \quad (506)$$

Considering that both  $A$  and  $a$  vary from zone to zone it is necessary, first of all, to perform the integration of (506) in zones. Moreover, in order to separate the variables it is necessary to resort to some form of approximation.

Various methods of solving (506), or forms of similar nature (i.e., based on zones), have been offered. They are all characterized by the establishment, among the variables of the chief equation itself, of certain relations which remain nearly constant for a given trajectory and whose approximate mean value for a given trajectory may be evaluated independently of the integration. The separation of these relations from the variables of the chief equation, in the form of mean-value constants, then leaves the equation integrable. However, these devices do not eliminate the necessity of performing the integration zone by zone, with the appropriate retardation function applied in each zone. Several of these methods have been developed to the extent of preparing tables of the integrated values of one or more forms employed in the final evaluation of the differential equations, and thus have eliminated the necessity of actually performing integrations zone by zone for each problem.\*

It is noteworthy that the type of procedure just discussed accepts approximation in connection with the form of the chief equation, but adheres without compromise to the best experimental determination of the retardation function that may be available. This is just the reverse of the earlier mode of attack that has been discussed in the foregoing article. Also, it is significant that the later procedure abandons the attempt to perform the necessary integrations for an entire trajectory in a single step, and resorts to separate integrations with respect to each of the several velocity zones that may be included within the limits of a given trajectory.

### Siacci's Method

506. As an illustration of methods of the type mentioned in the foregoing article we shall examine briefly *Siacci's Method*. Col. Siacci, an Italian artillery officer, published his method initially in 1880, and in 1888 and 1896 he published extensions of his method, including additional refinements and tables based on different retardation functions than first assumed. His method, while employing devices similar in character to those of several contemporary methods, achieved

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\* An exhaustive discussion of methods of the type noted here, as well as of the earlier types mentioned in article 504, is given in pp. 88-215, *Handbook of Ballistics*, Vol. I, by Cranz and Becker.

greater simplicity than did the other methods and was widely used by the ballisticians of nearly all countries, including the United States, until about the time of the World War. In the United States, Siacci's Method was further simplified in its application to actual solutions of the trajectory, although not altered in principle, by Col. Ingalls (U.S. Army). Siacci adapted his method not only to Mayevski's retardation functions but also to other retardation functions of the zone type. In the United States, Mayevski's functions in the form given by Ingalls (art. 409) are used with Siacci's Method.

507. One of the approximations made by Siacci was designed to avoid the complications that would result from introducing into the chief equation the function representing the change of air density with altitude (that is, a function such as  $H_v$ , which already has been defined in article 420). It consisted simply of assuming that this function can be reduced to a mean value for an entire trajectory, and in this manner be treated as a constant. Various procedures have been offered for determining such a mean air density; the practice in our services, in connection with Siacci's Method, has been to assume the mean density for the entire trajectory to be the density at the height of its mean ordinate, and to take the latter as equal to two-thirds of the height of the maximum ordinate (which is exact only for a true parabola). This approximation is found to be satisfactory for trajectories having angles of departure up to about  $15^\circ$ .\* In our services this altitude-density relation has been denoted by the symbol  $f_a$  and called the altitude factor. The factor  $f_a$  has been chosen to represent the ratio of air density at the surface to air density at the height of the mean ordinate, and hence it operates as a divisor of the surface density factor  $\delta$ , or as a multiplier in  $C$  (note that the ratio  $H_v$  has been taken in the reverse sense).

Since the value of  $f_a$  depends upon the maximum ordinate, and the determination of the latter, as well as of all other elements of a trajectory, depends upon prior knowledge of  $f_a$ , it is necessary in the Siacci Method to make a solution first without  $f_a$  and to determine an approximate value of this factor from the approximate maximum ordinate thus obtained. By means of further successive approximations the value of  $f_a$  may then be established with any desired degree of accuracy and the solution completed.

508. The principal feature of Siacci's Method is the device leading to the separation of the variables of the chief equation, for the purpose of putting the latter in shape for formal integration. This device consists of introducing into the chief equation a relation between the remaining velocity  $v$  and a component  $u$  of the latter obtained by projecting  $v$  vertically upon the line of departure (or a line parallel thereto). This component  $u$  is called the *pseudo velocity*. As shown by Figure 12, this relation is

$$u \cos \phi = v \cos \theta \quad (507)$$

also

$$v = \frac{u \cos \phi}{\cos \theta} \quad (508)$$

\* A somewhat closer approximation for the mean ordinate is given on page VIII, *Artillery Circular M* (1917); it shows that the assumption  $y_m = \frac{2}{3}y_a$  is in error by less than 1% for ordinary trajectories having angles of departure less than about  $15^\circ$ .

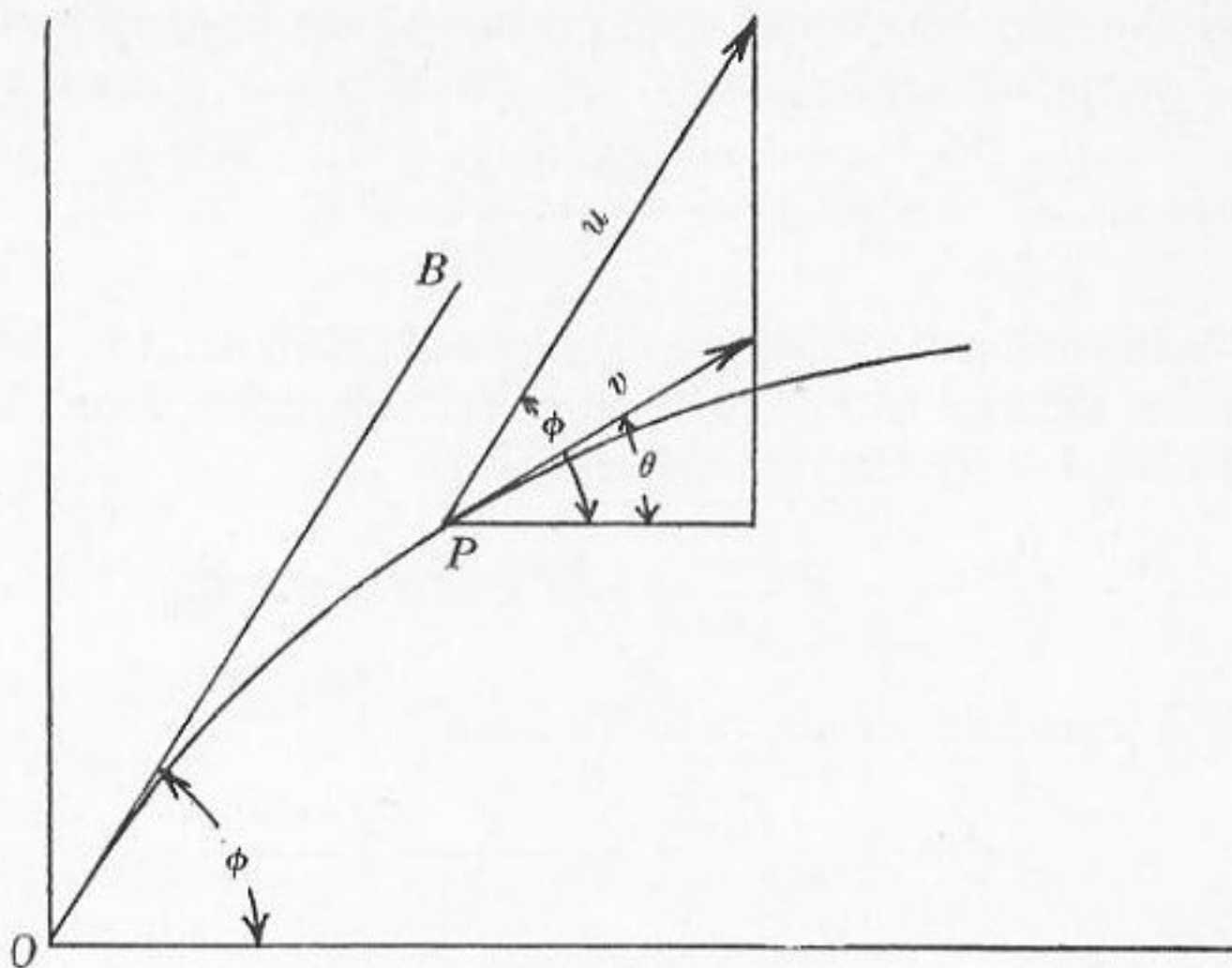


FIGURE 12

Substituting these equivalents of  $v \cos \theta$  and of  $v$  in the left-hand and right-hand members of (506), and introducing into the latter at the same time also the altitude factor  $f_a$ , we have

$$d(u \cos \phi) = \frac{1}{f_a C} \times \frac{A}{g} \left( \frac{u \cos \phi}{\cos \theta} \right)^{(a+1)} d\theta.$$

Writing  $du \cos \phi$  for  $d(u \cos \phi)$  in the above, and expanding the term in the bracket, we can transpose the equation to the form

$$f_a C \times \frac{g}{A} \times \frac{du}{u^{(a+1)}} = \frac{\cos^a \phi}{\cos^{(a+1)} \theta} d\theta. \quad (509)$$

The following operations are now applied to terms in the right-hand member of (509),

$$\begin{aligned} \cos^a \phi &= \cos^{(a-2)} \phi \cos^2 \phi \\ \frac{1}{\cos^{(a+1)} \theta} &= \frac{1}{\cos^{(a-1)} \theta} \sec^2 \theta \end{aligned}$$

and substituting these equivalents in (509) we have

$$f_a C \times \frac{g}{A} \times \frac{du}{u^{(a+1)}} = \left( \frac{\cos^{(a-2)} \phi}{\cos^{(a-1)} \theta} \right) \cos^2 \phi \sec^2 \theta d\theta. \quad (510)$$

509. Equation (510) has been put into the form shown above deliberately for the purpose of creating therein the expression  $\left( \frac{\cos^{(a-2)} \phi}{\cos^{(a-1)} \theta} \right)$ , since it has been

found that a mean value of this expression for an entire trajectory can be approximated satisfactorily in terms of initial elements, and hence can be determined in-

dependently of the variables  $\theta$  and  $u$  and hence of the integration of (510). We will therefore denote this expression by the special symbol  $\beta$ , so that

The factor  $\beta$  
$$\beta = \frac{\cos^{(a-2)} \phi}{\cos^{(a-1)} \theta} \quad (511)$$

and treat  $\beta$  as being independent of the variables of (510), i.e., as a constant.

We may now proceed to put (510) into form for integration, first replacing the value in the bracket by its symbol  $\beta$ , whence

$$f_a C \times \frac{g}{A} \times \frac{du}{u^{(a+1)}} = \beta \cos^2 \phi \sec^2 \theta d\theta$$

and we may then transpose the above to the form

$$\sec^2 \theta d\theta = \left( \frac{f_a C}{\beta} \times \frac{g}{A \cos^2 \phi} \right) \frac{du}{u^{(a+1)}}. \quad (512)$$

Since  $f_a$  and  $\beta$  are to be regarded as constants applying to an entire trajectory, we may simplify our notation by combining them with  $C$ , which already contains other such constants; thus

The Siacci  $C$ , or  $C_s$  
$$C_s = \frac{f_a}{\beta} C. \quad (513)$$

This artificial combination represented by  $C_s$  is called the *Siacci C*, and it is a special form of the ballistic coefficient to be used with Siacci's Method only.

We may then finally write the chief equation, from (512), in the form

The chief equation in terms of Mayevski's functions and Siacci  $C$  
$$\sec^2 \theta d\theta = \left( \frac{C_s g}{A \cos^2 \phi} \right) \frac{du}{u^{(a+1)}}. \quad (514)$$

Now all the terms in the bracket in the right-hand member of (514) are independent of the variables  $\theta$  and  $u$ , and the latter are completely separated. The integration may then be expressed as follows,

$$\int_{\phi}^{\theta} \sec^2 \theta d\theta = \frac{C_s g}{A \cos^2 \phi} \int_V^u \frac{du}{u^{(a+1)}} \quad (515)$$

the limits for  $\theta$  being its value  $\phi$  at the origin and its general value  $\theta$  at any point in the trajectory, and for  $u$  its value  $V$  at the origin and its general value  $u$  at any point in the trajectory.

The integration of (515) presents no difficulties except that for the right-hand member it must be performed in as many steps as there are velocity zones within the chosen limits of velocity. For example, if Mayevski's retardation functions are to be used, and if the integration is to be between the velocities 2900 f.s. and 1600 f.s., then the integration must proceed through the following steps: from 2900 f.s. to 2600 f.s. with  $a=1.55$  and  $\log A=7.60905-10$ ; from 2600 f.s. to 1800 f.s. with  $a=1.7$  and  $\log A=7.09620-10$ ; and from 1800 f.s. to 1600 f.s. with  $a=2$  and  $\log A=6.11926-10$  (art. 409).

510. We will now examine very briefly the manner of applying the solution of the chief equation to the solution of another of the differential equations (art. 503). The equation for range is

$$dx = -\frac{v^2}{g} d\theta. \quad (502)$$

From (508) we have the relation  $v = u \cos \phi \sec \theta$ ; substituting this in (502) we have

$$dx = -\left(\frac{u^2 \cos^2 \phi}{g}\right) \sec^2 \theta d\theta$$

and substituting in the above the value of  $\sec^2 \theta d\theta$  as given by (514) we have

$$dx = -\left(\frac{u^2 \cos^2 \phi}{g}\right) \left(\frac{C_s g}{A \cos^2 \phi}\right) \frac{du}{u^{(a+1)}}$$

which simplifies to

$$dx = -\frac{C_s}{A} \times \frac{du}{u^{(a+1)}}. \quad (516)$$

The integration of (516), between the origin where  $u = V$  and  $x = 0$ , and any point in the trajectory where velocity and range have the general values  $u$  and  $x$ , is then expressed as follows

Integration of the differential equation for range

$$\int_0^x dx = -\frac{C_s}{A} \int_V^u \frac{du}{u^{(a+1)}} \quad (517)$$

and this integration again offers no difficulties except that it must be performed in steps for each velocity zone within the limits of  $V$  and  $u$ , as already explained in the foregoing article.

A simplification of the ultimate process of solution for range may be achieved by tabulating the integration of the right-hand member of (517) for the entire range of velocities likely to be encountered in practice. This is done as follows. We will choose for the tabulation the quantity

The space function

$$S_u = -\frac{1}{A} \int \frac{du}{u^{(a+1)}} \quad (518)$$

and starting with 3600 f.s. perform the integration successively, for small intervals of velocity, down to the lowest velocity likely to be used, changing the values of  $A$  and  $a$  as we pass from zone to zone. Having once done this, the value of the integral for any velocity may be found directly from the table by entering with the given velocity, and the solution of (517) is simplified to

$$x = C_s(S_u - S_V). \quad (519)$$

511. The integral denoted by  $S_u$  is called the *space function* in Siacci's Method. In a similar manner an *inclination function*  $I_u$  may be separated from

(515) and tabulated. Operations similar to those shown above for the range equation (502), lead to an *altitude function*  $A_u$  when applied to (503), and to a *time function*  $T_u$  when applied to (504). Tables of each of these functions were prepared by Si-

acci. Ingalls prepared tables of the same functions, in English units, for velocities from 3600 f.s. to 100 f.s., using Mayevski's retardation expressions in the form given in article 409. Ingalls' tables appeared soon after the first publication of Siacci's Method and have been used by our services, in connection with the latter, since that time.

Tabulation of integral functions for Siacci's Method

512. As shown in article 509, the factor  $\beta$  was arbitrarily chosen to represent the relation

$$\beta = \frac{\cos^{(a-2)} \phi}{\cos^{(a-1)} \theta} \quad (511)$$

and then treated as a constant (contained in  $C_s$ ) in the chief equation (515). Since the cosine of a small angle varies but little from unity, it is evident that  $\beta$  does not vary much from unity for small values of  $\phi$  and  $\theta$ . For example, let us examine the values of  $\beta$  for the trajectory of the 16''2600 f.s. gun at about 13,300 yards, for which the angle of departure is about  $7^\circ$ , the angle of fall about  $9^\circ$ , and the striking velocity about 1821 f.s. Since the velocities on this trajectory are all in the zone 2600–1800 f.s., the value  $a=1.7$  applies at all points. We can then evaluate  $\beta$  for several points in the trajectory as follows. At the origin  $\theta = \phi = 7^\circ$ , whence

$$\beta = \frac{\cos^{-.3} 7^\circ}{\cos^{.7} 7^\circ} = 1.008.$$

At the summit  $\theta = 0^\circ$ , whence

$$\beta = \frac{\cos^{-.3} 7^\circ}{\cos^{.7} 0^\circ} = 1.002.$$

At the point in the descending branch where  $\theta = -\phi$

$$\beta = \frac{\cos^{-.3} 7^\circ}{\cos^{.7} 7^\circ} = 1.008.$$

At the point of fall  $\theta = \omega = 9^\circ$ , whence

$$\beta = \frac{\cos^{-.3} 7^\circ}{\cos^{.7} 9^\circ} = 1.011.$$

The flat portion of the trajectory on both sides of the vertex is relatively much longer than the more curved portions near the ends; the portion beyond the point in the descending branch where  $\theta = -\phi$  is only a small fraction of the whole length. A weighted mean of the above values is about 1.005, and this represents as accurately as necessary the mean value of  $\beta$  for the above trajectory. As the trajectory becomes more curved the mean value of  $\beta$  increases; for an angle of departure of  $15^\circ$  in the above case it becomes about 1.018. Siacci prepared fairly elaborate approximations of  $\beta$  for a wide variety of cases. The practice in our services has been to use  $\beta = \sqrt{\sec \phi}$ , which is found to give a satisfactory approximation for trajectories having angles of departure up to about  $15^\circ$ .

The significant feature in the choice of the factor  $\beta$  is that its value may be approximated from the value of  $\phi$ , which is known. The device which makes possible the establishment of this fortunate relation between  $\phi$  and  $\theta$  is the introduction of  $u$  as defined by (507).

513. Our present purpose is to investigate chiefly the character of Siacci's Method. The devices leading to the adoption of the approximate mean-value constants  $f_a$  and  $\beta$ , the introduction of these into the chief equation in such a manner as to render the latter subject to formal integration, the general character of the integration remaining to be performed, and the simplification of the latter (by means of tables of  $S_u$ ) in the case of the differential equation for range, have been examined. No new principles are involved in applying the process to the remaining differential equations; the choice of additional integral expressions for tabulation has been touched upon in article 511. It is not necessary for our present purpose to go further into the details of setting up the formulas and tables or of

making solutions. However, since Siacci's Method is still used, to a limited extent, by the U. S. Navy, a more complete treatment of the details leading to final solutions of the trajectory by this method is given in Appendix A.\*

Siacci's Method, receiving wide recognition almost immediately after its first publication, remained the standard among ballisticians of nearly all countries for some thirty-five years, until the time of the World War, when it fell quickly to a secondary position. The reason for this will be apparent from an examination of the limitations imposed by the approximations used in this method, which are characteristic of all similar methods.

514. The use of a mean altitude-density factor pertaining to an entire trajectory (such as  $f_a$ ) has the serious defect that it compromises the accuracy of all intermediate points of the trajectory in favor of the terminal point.

**Limitations of Siacci's Method** By resorting to sufficient elaboration in determining a mean ordinate, a satisfactory value of  $f_a$  may be determined even for trajectories having the greatest angles of departure used in practice, but any value of  $f_a$  so determined can satisfy the conditions for only one point. If it is desired to fix accurately a number of points in a trajectory, as is the case in antiaircraft problems, it is necessary either to accept a diminished degree of accuracy for all points other than the one for which  $f_a$  represents the correct mean, or else to determine a new  $f_a$  for each point to be found. Since accuracy at all points is essential in the computation of antiaircraft trajectories, the latter course becomes necessary in the application of Siacci's Method to such trajectories.

The same situation exists in the case of Siacci's other mean-value constant,  $\beta$ . The difficulties in determining a satisfactory value of this factor, even in the solution for terminal elements of trajectories having great angles of departure, are greater than in the case of  $f_a$ . The determination of satisfactory values of  $\beta$  for intermediate points, as required for antiaircraft trajectories, leads to further complications. The fortunate relation that  $\beta$  very nearly equals  $\sqrt{\sec \phi}$  ceases to be true not only for trajectories having angles of departure greater than about  $15^\circ$  but also for points other than the point of fall. In dealing with either of the latter cases it is necessary to resort to more elaborate approximations of the mean value of  $\beta$ .

The great favor accorded Siacci's Method rested solely upon the latter's simplicity. The development of long-range fire and antiaircraft fire, and the consequent need to deal with trajectories of great length and curvature, and with other than terminal elements, imposed on Siacci's Method such an elaborate superstructure of approximations that its feature of simplicity disappeared almost completely. The natural outcome of this was the abandonment of the entire system of approximations and the development of methods of unlimited application.

At the time of this writing Siacci's Method is still used by the U. S. Navy, in the computation of range tables for surface fire, for angles of departure up to  $15^\circ$ . It is anticipated, however, that in the near future the more recent method will be extended to the computation of our range tables for all angles of departure. Ballistic tables based on the latter method, which are at the time of this writing in the course of preparation by the U. S. Army, will reduce to very simple terms the immediate problem of constructing range tables.

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\* The complete derivation of all formulas of the Siacci Method and Ingalls' simplification thereof, and detailed demonstrations of their solution according to the latest practice in the U. S. Navy, are given in the 1926 and 1930 editions of this book. Ingalls' tables are given in the 1926 and 1930 editions of *Range and Ballistic Tables*.

THE SOLUTION OF THE TRAJECTORY IN AIR BY THE  
NUMERICAL INTEGRATION METHOD.

New Symbols Introduced

- $x', y'$  . . . . . Horizontal and vertical components, respectively, of the remaining velocity  $v$  of the projectile at any point  $x, y$ .
- $x'', y''$  . . . . . Horizontal and vertical components, respectively, of the total retardation experienced by the projectile at any point  $x, y$ .
- $x_0, y_0', v_0, E_0, \text{etc.}$  . . . . . } The subscripts denote the point to which the values pertain; for instance,  $x_0$  is the value of  $x$  at the point where the time is zero, i.e., at the origin;  $y_{1/4}'$  is the value of  $y'$  at the point which has been reached by the projectile after 1/4 second of flight;  $v_{3/2}$  is the value of  $v$  at the 1 1/2-second point;  $E_1$  is the value of  $E$  at the 1-second point; and so on.
- $x_{1/4}, y_{1/4}', v_{1/4}, E_{1/4}, \text{etc.}$  . . . . . }  
 $x_{3/2}, y_{3/2}', v_{3/2}, E_{3/2}, \text{etc.}$  . . . . . }
- $x_m'', x_m', y_m'', y_m'$  . . . . . The mean values of the retardation or velocity components for the particular interval in which the quantities are used.

601. The limitations of Siacci's Method and of other similar methods of approximation were appreciated long before these methods finally were relegated to a secondary position, during the period of the World War. That this did not happen much sooner may be attributed entirely to the additional labor incident to more accurate and universally applicable methods, and to the fact that the defects inherent in the approximation methods did not reach sufficiently great proportions, in connection with the comparatively restricted problems encountered prior to the World War, to warrant such additional labor.

It has long been known that the most complex of equations can be solved by a step-by-step process, and such procedure has been applied in astronomical computations for some two centuries. Euler, in 1753, gave a solution of this character for the ballistic problem. As described by Cranz in his *Handbook of Ballistics*, Vol. 1,

“His method depends on the summation of  $dx, dy, dt, ds$ . He treats the trajectory as a polygon of an infinite number of straight arcs  $\Delta s$ , and thence makes up the finite expression for the corresponding projections,  $\Delta x$  and  $\Delta y$ , as well as for the corresponding time  $\Delta t$ ; he then sums up the  $\Delta x, \Delta y, \Delta t$  to  $x, y, t$ . He assumes the quadratic law of air resistance.”

This description also fits the modern method remarkably well. Tables based on Euler's principle, and designed to eliminate the necessity for performing the tedious step-by-step integrations for each problem, appeared soon after this principle was first announced. The most noteworthy development along these

lines was given out by Otto, in 1842. In 1873, Bashforth published a method corresponding to that of Euler but based on the cubic law of air resistance.

Graphical equivalents of the step-by-step method of solution also have appeared from time to time. In 1828, Poncelet and Didion gave out a method which consisted essentially of constructing the trajectory graphically by a progression of arcs struck with successively determined values of the radius of curvature (which amounts practically to a step-by-step graphical construction according to the analysis given in article 501 and illustrated in Figure 11). Cranz, in 1896, proposed a graphical process that is a near equivalent to one of the computational methods now in favor. His process was based on short intervals of time, and depended on locating successive points along the trajectory by projecting forward along the tangent for the mean velocity during the particular interval, and vertically downward for the effect of gravity during the interval. In 1909, Cranz introduced a procedure which consisted essentially of graphing the differential equations (art. 503) and of making the necessary summations of  $dx$ ,  $dy$ ,  $dt$  by means of a planimeter or integraph.\*

602. It may then be said that the comparatively recent adoption of the step-by-step process as the ultimate in methods of solution of the ballistic problem, is a reversion to a principle that was among the very earliest to be applied to this problem,—although, since its reappearance in recent years, it has often been thought of and referred to as a new departure in the science of ballistics. In its modern form, the step-by-step solution of the trajectory is carried out by the process now generally known as *numerical integration*, differing from Euler's Method chiefly in that it is a higher development of the latter and in that more complete and more accurate retardation data are now available for use with it. Numerical integration has been applied extensively to the solution of differential equations since long before its recent reappearance in the field of ballistics, and the development of the process has resulted from its general applications rather than from its application in the latter field. Modern texts dealing with the solution of differential equations generally include treatments of the theory and practice of numerical integration.

Numerical integration methods for the solution of trajectories were developed independently and practically concurrently, both here and abroad, during the latter part of the World War. The French and British, however, led in the practical application of such methods and before the end of the war they were using ballistic tables based on numerical integration. In the United States the application of numerical integration to the solution of trajectories was due chiefly to F. R. Moulton, professor of astronomy at the University of Chicago. Moulton, having been commissioned a Major in the United States Army, was placed in charge of the Ballistics Branch of the Ordnance Department of that service early in 1918, and shortly thereafter gave out the essence of the methods which have since that time remained in favor in this country.†

603. Numerical integration consists essentially of evaluating a differential relation between two variables by assigning numerical values to one of them in

\* For further description and discussion of the methods here mentioned, and of many others, see *Handbook of Ballistics*, Vol. I, Cranz and Becker.

† In his book, *New Methods in Exterior Ballistics*, published in 1926, Moulton gives an account of his investigations in the field of ballistics both during the war period and subsequently in connection with his post-graduate instruction of officers of the U. S. Army and U. S. Navy at the University of Chicago.

Nature of numerical integration

successive, regular intervals, and finding the corresponding increments of the other for the chosen intervals. The integrated value of one of the variables with respect to any chosen limits of the other may be determined by summing the successive increments of the former within these limits. In applying numerical integration to the solution of the differential equations of the trajectory as they are stated in article 503, we are still confronted with the necessity of solving first the chief equation (505). Since the latter expresses a relation between  $v$  and  $\theta$ , we can either integrate  $v$  by assuming successive values of  $\theta$ , or integrate  $\theta$  by assuming successive values of  $v$ . That is, having the initial values at the origin, where  $\theta = \phi$  and  $v = V$ , we can assume successively decreasing values of  $\theta$ , say in  $10'$  intervals ( $\Delta\theta = 10'$ ), and determine the corresponding reductions in  $v$  (or  $\Delta v$ ); by reducing  $v$  successively according to the  $\Delta v$ 's thus found, we can establish the value of  $v$  corresponding to any value of  $\theta$ . Likewise, we can assume successively decreasing values of  $v$  (say  $\Delta v = 10$  f.s.), and thus determine the value of  $\theta$  corresponding to any value of  $v$ . Whichever of these procedures has been chosen, we can then apply a like process to each of the remaining differential equations. That is, if we have determined values of  $v$  corresponding to values of  $\theta$  (from (505)), we can proceed to integrate  $x$  by applying to (502) successive small changes in  $\theta$  and their corresponding changes in  $v$ . The variable chosen to be the one that is changed in successive regular intervals is called the *independent variable*.

Choice of independent variable

Solutions based on  $v$  as the independent variable are not convenient and are not used. The French have made extensive use of methods using  $\theta$  as the independent variable, and their ballistic tables (including some which are used in this text) are based on this method. A number of variations also have been introduced, both here and abroad, in which the independent variable is not a simple element of the trajectory but a variable arbitrarily arrived at by certain transformations of the fundamental equations (not, however, by resorting to approximation devices such as Siacci's). The method introduced by Moulton is based on the time  $t$  as independent variable, and this method is used by the U. S. Army for the computation of its numerical integration ballistic tables. Moulton's Method (sometimes referred to also as the rectangular method) is by far the most obvious of the many methods based on numerical integration. It possesses the advantages of directness, simplicity, and accuracy to a high degree, although it does not lend itself to labor saving technique as well as do some of the others.\* An elementary example of the application of numerical integration to the solution of a trajectory, using  $t$  as the independent variable, will be given presently. In the meantime we shall examine the general character of the process of numerical integration as applied to ballistics, by applying the process to some very simple cases.

604. Let us suppose that a projectile is fired practically horizontally, and that it is desired to determine how far the projectile will travel while its velocity is being reduced a given amount by air resistance. In order to confine our problem to simple terms, we shall assume a velocity reduction small enough to leave the path to be considered practically straight as well as horizontal—say a reduction from 1820 f.s. to 1780

Elementary solution with  $v$  as independent variable

\* Examples of solutions using  $\theta$  as independent variable, including a variation used by the French in the computation of their tables, are given in Chapters V and VI of *Ordnance Pamphlet No. 500* (U. S. Navy). For examples of solutions by Moulton's Method (using  $t$  as independent variable) and by Bennett's *tangent-reciprocal method* (based on an artificial variable) see *U. S. War Department Document No. 984* (1919).

f.s.—and also assume  $C=1$ . We shall now proceed to determine the distance covered by the projectile in steps each involving a reduction of 10 f.s., i.e., from 1820 to 1810, from 1810 to 1800, etc.

In the first of these steps the mean velocity of the projectile is 1815 f.s., and for this mean velocity the retardation according to the  $G$ -function is 435.35 f.s.s. (obtainable directly from Table I, since  $C=1$ ). The time elapsing during the reduction of 10 f.s. in velocity from 1820 f.s. to 1810 f.s. very nearly equals the

reduction in velocity divided by the retardation, or  $\frac{10 \text{ f.s.}}{435.35 \text{ f.s.s.}}$ . The distance

covered during the same reduction very nearly equals the time elapsed multiplied by the mean velocity for the interval, or

$$\frac{10 \text{ f.s.}}{435.35 \text{ f.s.s.}} \times 1815 \text{ f.s.} = 41.7 \text{ feet.}$$

Thus we have determined that while its velocity is being reduced from 1820 f.s. to 1810 f.s., the projectile travels 41.7 feet.

For the second interval the reduction is from 1810 f.s. to 1800 f.s., the mean velocity is 1805 f.s., the retardation corresponding to the latter is 431.25 f.s.s., and the distance covered is

$$\frac{10}{431.25} \times 1805 = 41.9 \text{ feet.}$$

Proceeding similarly for the remaining intervals we obtain the results tabulated below.

<u>Reduction in <math>v</math> (f.s.)</u>	<u>Distance covered (feet)</u>
1820–1810	41.7
1810–1800	41.9
1800–1790	42.0
1790–1780	42.2
	<hr/>
	Total 167.8

605. As to the accuracy of the above determination, the following observations may be made. The use of the arithmetic mean of the retardation pertaining to the beginning and end of each 10 f.s. interval to find the time for that interval, and of a similarly determined mean velocity to find the distance covered, both are

sources of inaccuracy, since the velocity-retardation relation is not a linear function. What degree of inaccuracy has been incurred by using 10 f.s. intervals can be determined very easily by making the computations for much smaller intervals. A determination of the distance covered for the velocity reduction from 1820 f.s. to 1810 f.s., in five 2 f.s. intervals, also gives 41.7 feet (in fact, the result obtained by using 2 f.s. intervals from 1820 f.s. to 1810 f.s. agrees to five places with that obtained in one step as above). Using but a single interval for the entire reduction from 1820 f.s. to 1780 f.s., the result obtained differs by less than a tenth of a foot from that obtained in four steps as above. (Taken to five places, the result obtained by using one 40 f.s. interval is 167.75 feet, compared to 167.78 feet as obtained by using four 10 f.s. intervals.)

It is apparent, then, that we can secure as great accuracy as may be desired, in this method, merely by regulating the size of the interval used in the inde-

pendent variable. In the specific instance defined by the above problem it appears that for the determination of distances, intervals of 10 f.s. afford accuracy to the hundredth of a foot, while for accuracy to the tenth of a foot intervals as great as 50 f.s. certainly might be used. In any case, the magnitude of the interval that may be used depends on the quantities involved and on the degree of accuracy desired, and it may be determined readily by a few trials. Inspection of the differences between the successive values obtained with any chosen interval gives a good indication of the degree of inaccuracy that is being incurred by the use of linear averages within the intervals. Inspection of the first differences in the tabulation given for the results in the above problem indicates at once that the inaccuracy due to the use of 10 f.s. intervals in this case is less than a tenth of a foot in each interval.

606. The above problem affords also a comparison, in very simple terms, between the process of determining elements of a trajectory by numerical integration and according to a tabulated retardation function (such as the  $G$ -function), and that of determining the same elements by formal integration and according to a retardation function of the character of Mayevski's. Siacci's Method embodies the

latter process. The solution, according to Siacci's Method, of the problem stated in the foregoing article, involves the solution of formula (519). Since we have assumed a trajectory that is practically horizontal and short enough to be considered a straight line, we have  $\phi = \theta = 0$ , whence  $\beta = 1$ ,  $f_a = 1$ , and also  $u = v$ . Then, since we also have assumed  $C = 1$ , the solution of (519) amounts

merely to the integration of Siacci's space function (518) between the limits 1820 f.s. and 1780 f.s. These velocities do not all lie within any one of Mayevski's zones, and the integration of (518) must therefore be performed in two parts, the first from 1820 f.s. to 1800 f.s. and the second from 1800 f.s. to 1780 f.s., using the proper values of  $a$  and  $A$  in each case (art. 409). Substituting the required values of  $a$  and  $A$ , the process is expressed as follows (the numbers in parentheses are  $\log A$ ).

$$x = -\frac{1}{(6.11926 - 10)} \int_{1800}^{1780} \frac{du}{u} + \frac{1}{(7.09620 - 10)} \int_{1820}^{1800} \frac{du}{u^{0.7}}$$

Performing these integrations, we have\*

$$x = (-56872 + 56957) + (-25308 + 25392) = 169 \text{ feet.}$$

The result thus obtained does not differ greatly from that arrived at by the step-by-step process. The difference found is due partly to the fact that Mayevski's retardation functions, while representing good average relations for whole zones, cannot define the retardation for each velocity as accurately as is possible with a tabular function, and partly to the fact that the  $G$ -function and Mayevski functions are based on different experimental data and hence involve real differences, apart from such as may arise from the manner in which the functions are expressed. Although in the simple problem here considered the labor involved in the solution by Siacci's Method does not differ greatly from that involved in the numerical integration process, it is to be noted that the identical amount of labor, by Siacci's method, is sufficient to solve similar problems for any velocities within the same two zones (i.e., any from 2600 f.s. to 1370 f.s.), while by numerical inte-

\* The integrals reduce, respectively, to  $(4.24296) \log u$  and  $(3.42668) u^{0.3}$ , the numbers in parentheses being logarithms. According to Ingalls' Tables, based on computations correct to at least seven places, the result of this integration is 168.9 feet.

gration the labor, of course, increases greatly for more widely separated velocities.

607. The use of  $v$  as the independent variable offered no difficulties in the above problem, because definite limits of velocity could be assigned to each interval independently of any other variables entering into the solution, and hence the mean velocity and mean retardation for each interval could be found directly. In the more general case,  $v$  is modified also by a component of gravity depending upon the angle of inclination  $\theta$ , while the latter, in turn, cannot be found without knowing  $v$ . In this case it is necessary to resort to successive approximations to establish the correct relations for each interval, and this applies whether the independent variable is  $v$  or  $\theta$ , or any other element; but the use of  $v$ , in the general case, is less convenient than the use of  $\theta$  or  $t$ . The character of the process that enters into the general case can be illustrated very simply as follows.

608. Let us suppose again that a projectile is fired practically horizontally, and that it is desired in this case to determine how far the projectile will travel during a given interval of time. In order to leave the problem in simple terms we shall again consider an interval short enough so that the portion of the trajectory considered will remain practically straight as well as horizontal,—and take  $C=1$ . Let us then determine the distance covered by the projectile in  $\frac{1}{2}$  second, its initial velocity being 1800 f.s., and  $C=1$ , and proceed with the computations in five intervals of one-tenth of a second each.

Elementary  
solution  
with  $t$  as  
independent  
variable

We know that the initial velocity is 1800 f.s., and from Table I we find that at this velocity the retardation (for  $C=1$ ) is 429.2 f.s.s. Then, as a first approximation, we can say that the reduction of velocity during the first tenth of a second is 42.92 f.s., and the remaining velocity at the end of this interval is  $1800 - 42.9 = 1757.1$  f.s. A more accurate determination of the reduction of velocity during this interval may now be obtained by finding the retardation corresponding to the *mean* velocity in the interval, which is practically  $\frac{1}{2}(1800 + 1757.1) = 1778.6$  f.s.

The corresponding retardation is 420.33 f.s.s., whence for one-tenth of a second the reduction in velocity is 42.03 f.s. A second approximation of the remaining velocity at the end of the interval is then  $1800 - 42.0 = 1758.0$  f.s., and of the *mean* velocity for the interval  $\frac{1}{2}(1800 + 1758.0) = 1779.0$  f.s. With the latter we now find a retardation of 420.49 f.s.s., whence the velocity reduction for one-tenth of a second is 42.05 f.s., and the remaining velocity at the end of the interval is  $1800 - 42.0 = 1758.0$  f.s., which agrees with the next preceding approximation. Since there has been no change since the next preceding determination of the velocity for the end of the interval, there can be no further change in the value of the *mean* velocity last found, and we can take the latter to be 1779.0 f.s. We have now established the mean velocity for the first interval to the nearest tenth of a foot second, which is sufficiently accurate for our purpose. The distance covered during the interval is then found by multiplying this mean velocity by the time, thus  $1779.0 \text{ f.s.} \times 0.1 \text{ sec.} = 177.9 \text{ feet.}$

Successive  
approximations

Proceeding similarly with the second interval of one-tenth of a second, we start with the final approximation of the remaining velocity already found for the end of the first interval, or 1758.0 f.s. The corresponding retardation is 412.0 f.s.s., the reduction of velocity for one-tenth of a second is 41.20 f.s., and the first approximation of the remaining velocity at the end of the second interval is  $1758.0 - 41.2 = 1716.8$  f.s., and of the mean velocity for this interval  $\frac{1}{2}(1758.0 + 1716.8) = 1737.4$  f.s. A second approximation of the retardation is now found to be 403.6 f.s.s., of the reduction in velocity 40.36 f.s., of the remaining velocity  $1758.0 - 40.4 = 1717.6$  f.s., and of the mean velocity  $\frac{1}{2}(1758.0 + 1717.6) = 1737.8$  f.s. No further

change will be found in the latter, and the distance for the second interval is then  $1737.8 \times 0.1 = 173.8$  feet. Carrying this process forward through the remaining three intervals, the results are as follows.\*

<u>Time interval (seconds)</u>	<u>Distance covered (feet)</u>
0-0.1	177.9
0.1-0.2	173.8
0.2-0.3	169.8
0.3-0.4	166.0
0.4-0.5	162.4
	Total 849.9

We also may tabulate the results more conveniently as follows, showing the remaining velocity, as well as the distance, at the end of each of the intervals.

<u>t (seconds)</u>	<u>v (f.s.)</u>	<u>x (feet)</u>
0	1800	0
0.1	1758.0	177.9
0.2	1717.6	351.7
0.3	1678.8	521.5
0.4	1641.5	687.5
0.5	1605.8	849.9

609. As to the accuracy of the above determination, it is found that by halving the size of the interval the results obtained do not differ from the above, while by doubling the interval the differences at the end of one-half second are about 0.1 f.s. in the velocity and about 0.2 feet in the distance. By computing the entire one-half second in a single interval, the differences noted are about 0.7 f.s. in the velocity and about 1.4 feet in the distance. Here again we find that by adjusting the size of the interval we can regulate the accuracy of the result according to our requirements.

The significant difference between the process used in the present case and that used in the case of the problem stated in article 604, is that in the present case it has been necessary to arrive at the mean retardation and mean velocity for each interval by a series of approximations, since the two are dependent upon each other. † In article 604, as already noted, this was unnecessary because the velocity limits of each interval were fixed definitely in advance.

610. We may now proceed to examine the essential features of the process of numerical integration as applied to the solution of a trajectory. The method that will be illustrated is based on the use of  $t$  as the independent variable and it is, fundamentally, the method that has been adopted by the U. S. Army for the computation of ballistic tables. For this method a very simple set of differential equations is set up as follows.

The retardation of a projectile due to air resistance, taking into account all known factors pertaining to the projectile itself and to the density of the air, is

\* The process can, of course, be condensed somewhat. For example, the mean velocity for any interval of one-tenth of a second may be found more directly by subtracting one-twentieth of the retardation (in f.s.s.) from the initial velocity for that interval.

† This problem could have been solved more simply by using  $v$  as the independent variable, as in art. 604, for it will be recalled that the latter method yielded a value of  $t$  for the end of each interval. The more complicated process has been used here purposely, because it illustrates in simple terms the general nature of successive approximations within an interval.

expressed by the formula

$$E = \frac{G_v \times H_v}{C} \quad (408) \quad (601)$$

which has already been explained in detail in article 429. Now just as the remaining velocity at any point in the trajectory has the horizontal and vertical components, respectively,  $v \cos \theta$  and  $v \sin \theta$ , the retardation due to air resistance has the components  $E \cos \theta$  and  $E \sin \theta$ . The retardation of the projectile in the horizontal plane is due entirely to the air resistance; in the vertical plane, however, the projectile's initial velocity is retarded both by air resistance and by gravity. Since acceleration (or, as it is in this case, retardation) is the second derivative of distance with respect to time, the horizontal component of retardation of the

projectile is defined by  $\frac{d^2x}{dt^2}$  and the vertical component by  $\frac{d^2y}{dt^2}$ . We may then

write for these retardations, including in the case of the vertical component the effect of gravity,

Differential equations of trajectory for solution with  $t$  as independent variable

$$\frac{d^2x}{dt^2} = -E \cos \theta \quad (602)$$

$$\frac{d^2y}{dt^2} = -E \sin \theta - g. \quad (603)$$

Equations (601), (602), and (603) form the basis of the method of solution which is to be illustrated. In order to simplify the notation to be used in carrying out computations, we shall adopt the following.

$x$  = horizontal distance, or range, in feet.

$y$  = vertical distance, or ordinate, in feet.

$x'$  = horizontal velocity component, in f.s.

$y'$  = vertical velocity component, in f.s.

$x''$  = horizontal retardation component, in f.s.s.

$y''$  = vertical retardation component, in f.s.s.

In accordance with the above notation the entire system of formulas required in the solution may now be written as follows:

$$E = \frac{G_v \times H_v}{C} \quad C = \frac{w}{\delta i d^2} \quad (601)$$

$$x'' = -E \cos \theta \quad (602)$$

$$y'' = -E \sin \theta - g \quad (603)$$

$$x' = v \cos \theta \quad \text{or} \quad \cos \theta = \frac{x'}{v} \quad (604)$$

$$y' = v \sin \theta \quad \text{or} \quad \sin \theta = \frac{y'}{v} \quad (605)$$

$$v = \sqrt{(x')^2 + (y')^2}. \quad (606)$$

Also, in order to distinguish among values of the various elements at the various points through which the solution is to proceed, time subscripts will be used to

denote the point to which any value applies. For example,  $E_0, x_0, x_0', x_0'', y_0, y_0', y_0'', v_0$ , etc., will denote values at the point where  $t=0$ , i.e., at the origin;  $E_{1/4}, x_{1/4}, x_{1/4}', x_{1/4}'', y_{1/4}, y_{1/4}', y_{1/4}'', v_{1/4}$ , etc., will denote values at the point where  $t = \frac{1}{4}$  second; and so on. The notations  $x_m', x_m'', y_m'$ , and  $y_m''$ , will be used to denote the mean values of  $x', x'', y'$ , and  $y''$ , respectively, for a given interval.

Elementary example of solution of trajectory, with  $t$  as independent variable

611. For illustration of the process of solution we will compute several points of the trajectory of the 16'' 2600 f.s. gun for the angle of departure  $30^\circ$ . The  $G$ - and  $H$ -tables are appended to this chapter for convenience in following this solution. The necessary initial data follow:

$$V = 2600 \text{ f.s.}, \quad \phi = 30^\circ, \quad d = 16'', \quad w = 2100 \text{ lbs.}, \quad i = .61, \quad \delta = 1.00$$

whence we have

$$C = \frac{w}{\delta i d^2} \quad (406)$$

$\delta = 1.00$	log 0.00000	colog 0.00000
$w = 2100$	log 3.32222	
$i = .61$	log 9.78533 - 10	colog 0.21467
$d^2 = 256$	log 2.40824	colog 7.59176 - 10
<hr/>		
$C =$	log 1.12865	

and this value of  $C$  will not change during the computation.

From (604) and (605) we have  $x' = v \cos \theta$  and  $y' = v \sin \theta$ , and since at the origin  $\theta = \phi$  and  $v = V$ , we may compute the horizontal and vertical velocity components at the origin.

$v = V = 2600$	log 3.41497	log 3.41497
$\theta = \phi = 30^\circ$	lcos 9.93753 - 10	lsin 9.69897 - 10
<hr/>		
$x_0' = 2251.6$	log 3.35250	
<hr/>		
$y_0' = 1300.0$	log 3.11394	

Elements at the origin

Likewise we may compute the horizontal and vertical acceleration components at the origin, using (601), (602) and (603). At the origin  $v = V = 2600$ , hence we look up  $G_v$  in the  $G$ -table for the velocity 2600 f.s.; we find that it is 769.3 f.s. At the origin  $y = 0$  and hence  $H_y = 1$ , or  $\log H_y = 0.00000$ , which we find also from the  $H$ -table for  $y = 0$ . Since  $\theta = \phi = 30^\circ$ , and  $C$  has already been computed, we have all the data required to solve (601) (602) and (603).

$G_v = 769.3$	log 2.88610	
$H_y$	log 0.00000	
$C$	log 1.12865	colog 8.87135 - 10
<hr/>		
$E_0$	log 1.75745	
<hr/>		
$E_0$	log 1.75745	log 1.75745
$\theta = \phi = 30^\circ$	lcos 9.93753 - 10	lsin 9.69897 - 10
<hr/>		
$x_0'' = 49.54 \text{ f.s.s.*}$	log 1.69498	
<hr/>		
$E \sin \theta = 28.60 \text{ f.s.s.}$	log 1.45642	
$g = 32.16 \text{ f.s.s.}$		
<hr/>		
$y_0'' = 60.76 \text{ f.s.s.*}$		

\* The minus signs are to be understood in applying these values.

FIRST INTERVAL

612 We now know that at the origin the horizontal velocity is  $x' = 2251.6$  f.s. and that it is decreasing at this point at the rate of  $x'' = 49.54$  f.s.s.; also that the vertical velocity is  $y' = 1300$  f.s. and that it is decreasing at the rate of  $y'' = 60.76$  f.s.s. If we assume a small interval of time for instance  $\frac{1}{4}$  second, we can estimate what the velocity components will be at the end of that interval; they will be, approximately, the original values minus one fourth of the retardation for one second.

We can then make the predictions,

First Prediction

$$x_{\frac{1}{4}}' = x_0' - \frac{1}{4}x_0'' = 2251.6 - \frac{49.54}{4} = \underline{2239.2 \text{ f.s.}}$$

$$y_{\frac{1}{4}}' = y_0' - \frac{1}{4}y_0'' = 1300.0 - \frac{60.76}{4} = \underline{1284.8 \text{ f.s.}}$$

$$v_{\frac{1}{4}} = \sqrt{(x_{\frac{1}{4}}')^2 + (y_{\frac{1}{4}}')^2} = \underline{2581.6 \text{ f.s.}}$$

$x_{\frac{1}{4}}'$	= 2239.2	log 3.35009,	2log 6.70018,	$(x_{\frac{1}{4}}')^2$	= 5014000
$y_{\frac{1}{4}}'$	= 1284.8	log 3.10883,	2log 6.21766,	$(y_{\frac{1}{4}}')^2$	= 1650700
$v_{\frac{1}{4}}^2$	=	.....	.....		6664700
6,664,700,	log 6.82378,	$\frac{1}{2}$ log 3.41189,		$v_{\frac{1}{4}}$	= 2581.6

and from (604) and (605),

$x_{\frac{1}{4}}'$	= 2239.2	.....	log 3.35009		
$y_{\frac{1}{4}}'$	= 1284.8	.....	.....	log 3.10883	
$v_{\frac{1}{4}}$	= 2581.6	log 3.41189.	colog 6.58811 - 10	.....	colog 6.58811 - 10
$\theta_{\frac{1}{4}}$	= 29° 50' 46''	.....	lcos 9.93820 - 10	.....	lsin 9.69694 - 10

These predictions would be accurate only if the retardation had remained constant throughout the  $\frac{1}{4}$ -second interval, which we know is not the case. A more nearly correct value of the retardation may be obtained by taking the mean of the values for the beginning and end of the interval. We have already found what the retardation components are for the beginning of the interval; for the end of the interval we must solve for  $x_{\frac{1}{4}}''$  and  $y_{\frac{1}{4}}''$ , and to do this we will have to use the approximate data we have already found for the end of the interval.

Second Prediction

613. The approximate mean vertical velocity for the interval is evidently the mean of  $y_0'$  and  $y_{\frac{1}{4}}'$ , whence,

$$y_m' = \frac{1}{2}(y_0' + y_{\frac{1}{4}}') = \frac{1}{2}(1300.0 + 1284.8) = \frac{1}{2} \times 2584.8 = \underline{1292.4 \text{ f.s.}}$$

and the approximate height of the projectile at the end of the interval is given by the product of the mean vertical velocity and the time interval, whence,

$$y_{\frac{1}{4}} = \frac{1}{4} \times 1292.4 = \underline{323.1 \text{ feet}}$$

Entering the  $H$ -table with 323 feet we find  $\log H_y = 9.99557 - 10$ . And with the velocity we found existing at the end of the interval, 2581.6 f.s., we enter the  $G$ -table and find  $G_v = 760.7$ . We may now solve (601), (602) and (603) for  $x_{\frac{1}{4}}''$  and  $y_{\frac{1}{4}}''$ .

$G_v = 760.7$	.....	log 2.88121			
$H_y$	.....	log 9.99557 - 10			
$C$	.....	colog 8.87135 - 10			
$E_{\frac{1}{4}}$	.....	log 1.74813	.....	log 1.74813	
$\theta_{\frac{1}{4}} = 29^\circ 50' 46''$	.....	lcos 9.93821 - 10	.....	lsin 9.69694 - 10	
$x_{\frac{1}{4}}'' = 48.57 \text{ f.s.s.}$	.....	log 1.68634	.....		
	27.87	.....	log 1.44507		
	32.16				
$y_{\frac{1}{4}}'' = 60.03 \text{ f.s.s.}$					

We now have the retardation components for the beginning of the interval ( $x_0''$  and  $y_0''$ ) and also the approximate values for the end of the interval ( $x_{\frac{1}{4}}''$  and  $y_{\frac{1}{4}}''$ ), and we may proceed to correct our first estimate of the velocity components at the end of the interval.

The mean retardation components for the interval are evidently,

$$x_m'' = \frac{1}{2}(x_0'' + x_{1/4}'') = \frac{1}{2}(49.54 + 48.57) = \underline{49.06 \text{ f.s.s.}}$$

$$y_m'' = \frac{1}{2}(y_0'' + y_{1/4}'') = \frac{1}{2}(60.76 + 60.03) = \underline{60.40 \text{ f.s.s.}}$$

and we may then predict that the velocity components at the end of the  $\frac{1}{4}$ -second interval will be

$$x_{1/4}' = x_0' - \frac{1}{4}x_m'' = 2251.6 - \frac{1}{4} \times 49.06 = \underline{2239.3 \text{ f.s.}}$$

$$y_{1/4}' = y_0' - \frac{1}{4}y_m'' = 1300.0 - \frac{1}{4} \times 60.40 = \underline{1284.9 \text{ f.s.}}$$

For  $v$  we have, as before,

$$v_{1/4} = \sqrt{(x_{1/4}')^2 + (y_{1/4}')^2} = \underline{2581.8 \text{ f.s.}}$$

1284.9 f.s.

$$x_{1/4}' = 2239.3 \dots \log 3.35011 \dots 2 \log 6.70022, (x_{1/4}')^2 = 5,014,400$$

$$y_{1/4}' = 1284.9 \dots \log 3.10887 \dots 2 \log 6.21774, (y_{1/4}')^2 = 1,651,000$$

$$(v_{1/4}')^2 \dots \dots \dots 6,665,400$$

$$6,665,400, \log 6.82383, \frac{1}{2} \log 3.41192, \dots \dots \dots v_{1/4} = 2581.8 \text{ f.s.}$$

For  $\theta$  we have, using (604) and (605) as before,

$$x_{1/4}' = 2239.3 \dots \dots \dots \log 3.35011$$

$$y_{1/4}' = 1284.9 \dots \dots \dots \log 3.10887$$

$$v_{1/4} = 2581.8 \dots \dots \log 3.41192 \dots \dots \text{colog } 6.58808 - 10 \dots \dots \text{colog } 6.58808 - 10$$

$$\theta_{1/4} = 29^\circ 50' 44'' \dots \dots \text{lcos } 9.93819 - 10 \dots \dots \text{lsin } 9.69695 - 10$$

Comparing the results of the second prediction with those of the first we find very close agreement in all of the elements found. The closeness of the first prediction may be attributed to the very small time interval used. Had the time interval been larger we would probably have been obliged to make several predictions before coming close to the correct result.

**Limit of accuracy** It is customary to carry on the process until two successive sets of values of the retardation components agree to the hundredth of a foot-second-second. With five-place tables we could carry the process on for accuracy to the thousandth of a foot-second-second, but the degree of accuracy obtained by two decimal places in the retardation components is quite sufficient for any purpose. For velocity components, and for  $x$  and  $y$ , one decimal is sufficient.

We will make one more prediction in order to be certain that there will be no further change in the computed elements.

### Third Prediction

614. The corrected estimate of the mean vertical velocity becomes,

$$y_m' = \frac{1}{2}(1300.0 + 1284.9) = \underline{1292.4 \text{ f.s.}}$$

and hence the height of the projectile at the end of the interval becomes,

$$y_{1/4} = \frac{1}{4} \times 1292.4 = \underline{323.1 \text{ feet}}$$

From the  $H$ -table we find, with 323 feet, that  $\log H_y = 9.99557 - 10$ . And with the previously found velocity of 2581.8 f.s. we find in the  $G$ -table that  $G_v = 760.8$ .

The data for solving (601), (602), and (603), are now complete, and we proceed as before,

$$G_v = 760.8 \dots \dots \log 2.88127$$

$$H_y \dots \dots \log 9.99557 - 10$$

$$C \dots \dots \text{colog } 8.87135 - 10$$

$$E_{1/4} \dots \dots \log 1.74819 \dots \dots \log 1.74819$$

$$\theta_{1/4} \dots \dots \text{lcos } 9.93821 - 10 \dots \dots \text{lsin } 9.69693 - 10$$

$$x_{1/4}'' = 48.57 \text{ f.s.s.} \dots \dots \log 1.68640$$

$$\underline{27.87} \dots \dots \log 1.44512$$

$$\underline{32.16}$$

$$\underline{y_{1/4}'' = 60.03 \text{ f.s.s.}}$$

615. We now find exact agreement between the last two successive predictions of  $x''$  and  $y''$ . We may be certain, then, that any further continuation of the predictions will be merely a repetition of previous work, and that no further accuracy within the chosen limits may be obtained.

We complete the operation for the first interval by finding  $x$ . The horizontal velocity at the origin ( $x_0'$ ) was 2251.6 f.s.; at the end of  $\frac{1}{4}$  second it was  $x_{\frac{1}{4}}' = 2239.3$  f.s., this being the value from the last prediction. Hence the mean horizontal velocity for the interval was,

$$x_m' = \frac{1}{2}(2251.6 + 2239.3) = 2245.4 \text{ f.s.}$$

and the horizontal distance traveled by the projectile in  $\frac{1}{4}$  second was,

$$x_{\frac{1}{4}} = \frac{1}{4} \times 2245.4 = 561.4 \text{ feet}$$

SECOND INTERVAL

616. The entire process of locating the position of the projectile at the end of an interval of time of  $\frac{1}{4}$  second has been illustrated in the preceding articles. Before proceeding further it will be of advantage to arrange the information already obtained in a systematic manner, as follows:

$$V = 2600 \text{ f.s.} \qquad \phi = 30^\circ \qquad \log C = 1.12865$$

$t$	$v$	$\theta$	$x$	$x'$	$x''$	$\Delta x''$	$y$	$y'$	$y''$	$\Delta y''$
0	2600.0	30° 00'.0	0	2251.6	49.54	....	0	1300.0	60.76	....
$\frac{1}{4}$	2581.8	29° 50'.7	561.4	2239.3	48.57	0.97	323.1	1284.9	60.03	0.73
$\frac{1}{2}$	(2564.0)	(29° 41'.5)	.....	(2227.3)	(47.60)	(0.97)	(642.5)	(1270.0)	(59.30)	(0.73)

First Prediction

We may now proceed to a second  $\frac{1}{4}$ -second interval, starting from the point already found, and we will then have the location of the projectile at the end of  $\frac{1}{2}$  second. But our first prediction for the new interval may be made on the basis of information gained in the first interval. For

**Making use of the first differences** instance, from our tabulation above we find that  $x''$  decreased 0.97 f.s.s. in the first  $\frac{1}{4}$  second, and that  $y''$  decreased 0.73 f.s.s.; these quantities are the *first differences* in  $x''$  and  $y''$  columns, and they have been denoted by  $\Delta x''$  and  $\Delta y''$ . Assuming these changes to remain constant for the next interval we may predict that at the end of another  $\frac{1}{4}$  second we will have  $x_{\frac{1}{2}}'' = (48.57 - 0.97) = 47.60$ , and  $y_{\frac{1}{2}}'' = (60.03 - 0.73) = 59.30$ . These predicted values have been entered in the above tabulation in brackets.

The approximate *mean values* of  $x''$  and  $y''$  for the interval  $\frac{1}{4} - \frac{1}{2}$  second will evidently be,

$$x_m'' = \frac{1}{2}(48.57 + 47.60) = 48.08 \text{ f.s.s.}$$

$$y_m'' = \frac{1}{2}(60.03 + 59.30) = 59.66 \text{ f.s.s.}$$

and the approximate values of  $x'$  and  $y'$  at the end of the interval will be,

$$x_{\frac{1}{2}}' = 2239.3 - \frac{1}{4} \times 48.08 = 2227.3 \text{ f.s.}$$

$$y_{\frac{1}{2}}' = 1284.9 - \frac{1}{4} \times 59.66 = 1270.0 \text{ f.s.}$$

Likewise the approximate *mean value* of the vertical velocity for the new interval will be,

$$y_m' = \frac{1}{2}(1284.9 + 1270.0) = 1277.4 \text{ f.s.}$$

and the vertical distance traveled by the projectile during the  $\frac{1}{4}$  second will be,

$$\Delta y = \frac{1}{4} \times 1277.4 = 319.4 \text{ feet}$$

and hence its height at the end of the new interval will be,

$$y_{\frac{1}{2}} = y_{\frac{1}{4}} + 319.4 = 323.1 + 319.4 = 642.5 \text{ feet}$$

For  $v$  and  $\theta$  we use (604), (605) and (606) as usual. For convenience we will drop the time subscripts except for the quantity we are finding.

$$x' = 2227.3, \log 3.34778, 2\log 6.69556, (x')^2 = 4,960,900$$

$$y' = 1270.0, \log 3.10380, 2\log 6.20760, (y')^2 = 1,612,900$$

$$v_{\frac{1}{2}}^2 \dots \dots \dots 6,573,800$$

$$6,573,800, \log 6.81782, \frac{1}{2}\log 3.40891, v_{\frac{1}{2}} = 2564.0 \text{ f.s.}$$

$$\begin{array}{r}
 x' = 2227.3 \dots \log 3.34778 \\
 y' = 1270.0 \dots \log 3.10380 \\
 v = 2564.0 \dots \log 3.40891 \dots \text{colog } 6.59109 - 10 \dots \text{colog } 6.59109 - 10 \\
 \hline
 \theta_{1/2} = 29^\circ 41' 27'' \dots \text{lcos } 9.93887 - 10 \dots \text{lsin } 9.69489 - 10
 \end{array}$$

The approximate values we have found constitute the *first predictions* for the  $\frac{1}{2}$ -second point, and they have been entered in the above tabulation in brackets. We may proceed to correct these approximate values just as we did in the first interval.

### Second Prediction

617. With  $y = 642$  we find from the  $H$ -table that  $\log H_y = 9.99119 - 10$ ; and with  $v = 2564.0$  we find from the  $G$ -table that  $G_v = 752.4$ , whence we find from (601), (602) and (603),

$$\begin{array}{r}
 G_v = 752.4 \dots \log 2.87645 \\
 H_y \dots \log 9.99119 - 10 \\
 C \dots \text{colog } 8.87135 - 10 \\
 E_{1/2} \dots \log 1.73899 \dots \log 1.73899 \\
 \theta_{1/2} = 29^\circ 41' 27'' \dots \text{lcos } 9.93888 - 10 \dots \text{lsin } 9.69489 - 10 \\
 \hline
 x_{1/2}'' = 47.63 \text{ f.ss.} \dots \log 1.67787 \\
 \quad 27.16 \dots \log 1.43388 \\
 \quad 32.16 \\
 \hline
 y_{1/2}'' = 59.32 \text{ f.s.s}
 \end{array}$$

We will now correct all of the remaining values, and since the process is exactly the same as in the preceding article no further explanations are required.

$$\begin{array}{r}
 x_m'' = \frac{1}{2}(48.57 + 47.63) = \underline{48.10 \text{ f.s.s.}} \\
 y_m'' = \frac{1}{2}(60.03 + 59.32) = \underline{59.68 \text{ f.s.s.}} \\
 x_{1/2}' = 2239.3 - \frac{1}{4} \times 48.10 = \underline{2227.3 \text{ f.s.}} \\
 y_{1/2}' = 1284.9 - \frac{1}{4} \times 59.68 = \underline{1270.0 \text{ f.s.}} \\
 y_m' = \frac{1}{2}(1284.9 + 1270.0) = \underline{1277.4 \text{ f.s.}} \\
 \Delta y = \frac{1}{4} \times 1277.4 = \underline{319.4 \text{ ft.}} \\
 y_{1/2} = 323.1 + 319.4 = \underline{642.5 \text{ ft.}}
 \end{array}$$

Since the values of  $x_{1/2}'$  and  $y_{1/2}'$  have not changed from the previous prediction,  $v_{1/2}$  and  $\theta_{1/2}$  also will not change, and hence, from the first prediction,

$$v_{1/2} = \underline{2564.0 \text{ f.s.}}, \quad \theta_{1/2} = \underline{29^\circ 41' 27''}$$

It will be observed that the values obtained in the first and second predictions agree within the required limits of accuracy for the elements  $x'$ ,  $y'$ ,  $y$ ,  $v$  and  $\theta$ , and hence we have reached the limit of accuracy for these elements. The second predictions of  $x''$  and  $y''$  vary considerably from the first predictions and it would appear that we should make a further prediction for these elements. However, none of the quantities entering into the computation of  $x''$  and  $y''$  (by (601), (602) and (603)) have changed since the previous prediction, and hence the results of any further prediction could not possibly be different from the results of the previous prediction. It follows that we have also reached the limit of accuracy for  $x''$  and  $y''$ .

We may now complete the work of the second interval by computing the value of  $x$  for the  $\frac{1}{2}$ -second point. Proceeding as in article 615 (par. 2),

$$x_m' = \frac{1}{2}(2239.3 + 2227.3) = \underline{2233.3 \text{ f.s.}}$$

whence the increase in  $x$  during the interval  $\frac{1}{4} - \frac{1}{2}$  second was,

$$\Delta x = \frac{1}{4} \times 2233.3 = \underline{558.3 \text{ feet}}$$

and the value of  $x$  at the end of  $\frac{1}{2}$  second becomes,

$$x_{1/2} = 561.4 + 558.3 = \underline{1119.7 \text{ feet}}$$

It will be noted that there would be no purpose in making preliminary predictions for  $x$ , since  $x$  does not enter into the other quantities. We must make successive predictions of  $y$ ,  $\theta$ ,  $v$ , etc., because the other elements depend upon these. In the case of  $x$  we may make our final and only computation after the final values of the other elements have been found.

THIRD INTERVAL

618. We now tabulate our data once more, as follows:

$t$	$v$	$\theta$	$x$	$x'$	$x''$	$\Delta_1$	$\Delta_2$	$y$	$y'$	$y''$	$\Delta_1$	$\Delta_2$
0	2600.0	30° 00'.0	0	2251.6	49.54	.....	.....	0	1300.0	60.76	.....	.....
¼	2581.9	29 50'.7	561.4	2239.3	48.57	0.97	.....	323.1	1284.9	60.03	0.73	.....
½	2564.0	29 41'.4	1119.7	2227.3	47.63	0.94	0.03	642.5	1270.0	59.32	0.71	0.02
¾	(2546.4)	(29 32'.2)	.....	(2215.5)	(46.72)	(0.91)	(0.03)	(958.1)	(1255.3)	(58.63)	(0.69)	(0.02)

First Prediction

Having three quantities in the  $x''$  and  $y''$  columns, we are able to find the *first and second differences for  $x''$  and  $y''$* , i.e., *the change and rate of change in  $x''$  and  $y''$  for ¼-second intervals*. The first and second differences have been denoted by  $\Delta_1$  and  $\Delta_2$ , respectively. Assuming now that the *rates of change* (i.e.,  $\Delta_2$ ) remain the same for the next ¼ second, we predict the *changes* (i.e.,  $\Delta_1$ ) during the next ¼ second will be  $(0.94-0.03)=0.91$  for  $x''$ , and  $(0.71-0.02)=0.69$  for  $y''$ , and these predicted values have been entered in the above tabulation in brackets. We proceed to predict the values of  $x'$  and  $y'$  from these changes to be  $x_{¾}' = (47.63-0.91)=46.72$ , and  $y_{¾}' = (59.32-0.69)=58.63$ , and these predicted values have also been entered in brackets.\*

The first prediction is completed just as in article 616.

$$\begin{aligned}
 x_m'' &= \frac{1}{2}(47.63+46.72) = \underline{47.18 \text{ f.s.s.}} \\
 y_m'' &= \frac{1}{2}(59.32+58.63) = \underline{58.98 \text{ f.s.s.}} \\
 x_{¾}' &= 2227.3 - \frac{1}{4} \times 47.18 = \underline{2215.5 \text{ f.s.}} \\
 y_{¾}' &= 1270.0 - \frac{1}{4} \times 58.98 = \underline{1255.3 \text{ f.s.}} \\
 y_m' &= \frac{1}{2}(1270.0+1255.3) = \underline{1262.6 \text{ f.s.}} \\
 \Delta y &= \frac{1}{4} \times 1262.6 = \underline{315.6 \text{ ft.}} \\
 y_{¾} &= 642.5 + 315.6 = \underline{958.1 \text{ ft.}}
 \end{aligned}$$

$$\begin{aligned}
 x' &= 2215.5, \log 3.34547, 2\log 6.69094, (x')^2 = 4,908,400 \\
 y' &= 1255.3, \log 3.09875, 2\log 6.19750, (y')^2 = 1,575,800
 \end{aligned}$$

$$\begin{aligned}
 v_{¾}^2 & \dots \dots \dots 6,484,200 \\
 6,484,200, \log 6.81185, \frac{1}{2}\log 3.40592, v_{¾} &= \underline{2546.4 \text{ f.s.}}
 \end{aligned}$$

$$\begin{aligned}
 x' &= 2215.5 \dots \dots \dots \log 3.34547 \\
 y' &= 1255.3 \dots \dots \dots \log 3.09875 \\
 v &= 2546.4, \log 3.40592 \dots \dots \text{colog } 6.59408 - 10 \dots \dots \text{colog } 6.59408 - 10 \\
 \theta_{¾} &= 29^\circ 32' 11'' \dots \dots \text{lcos } 9.93955 - 10 \dots \dots \text{lsin } 9.69283 - 10
 \end{aligned}$$

Second Prediction

Proceeding as in article 617, we find from the  $H$ -table, with  $y=958$  feet, that  $\log H_y=9.98686-10$ , and from the  $G$ -table, with  $v=2546.4$  f.s., that  $G_v=744.4$ .

$$\begin{aligned}
 G_v &= 744.4 \dots \dots \dots \log 2.87181 \\
 H_y & \dots \dots \dots \log 9.98686 - 10 \\
 C & \dots \dots \dots \text{colog } 8.87135 - 10 \\
 E_{¾} & \dots \dots \dots \log 1.73002 \dots \dots \log 1.73002 \\
 \theta_{¾} &= 29^\circ 32' 11'' \dots \dots \text{lcos } 9.93954 - 10 \dots \dots \text{lsin } 9.69283 - 10 \\
 x_{¾}'' &= 46.73 \text{ f.s.s.} \dots \dots \dots \log 1.66956 \\
 26.48 & \dots \dots \dots \log 1.42285 \\
 \underline{32.16} & \\
 y_{¾}'' &= \underline{58.64 \text{ f.s.s}}
 \end{aligned}$$

\* See article 622.

The value of  $y_{3/4}''$  from the second prediction agrees exactly with that from the first; in the case of  $x_{3/4}''$  a difference of .01 f.s.s. will be noted. It is apparent that a difference of .01 f.s.s. in  $x''$  will cause a difference of only .0025 f.s. in  $x'$  during the  $\frac{1}{4}$ -second interval, which is beyond the limits we have chosen. Hence our first predictions were correct for all elements except  $x''$  and for the latter the second prediction is correct.

To find  $x_{3/4}$  we have,

$$\begin{aligned} x_m' &= \frac{1}{2}(2227.3 + 2215.5) = 2221.4 \text{ f.s.} \\ \Delta x &= \frac{1}{4} \times 2221.4 = 555.4 \text{ feet} \\ x_{3/4} &= 1119.7 + 555.4 = 1675.1 \text{ feet} \end{aligned}$$

FOURTH INTERVAL

619. For the next  $\frac{1}{4}$ -second interval we will proceed exactly as in the previous interval, and will carry out the process without interposing any further explanation.

$t$	$v$	$\theta$	$z$	$x'$	$x''$	$\Delta_1$	$\Delta_2$	$y$	$y'$	$y''$	$\Delta_1$	$\Delta_2$
0	2600.0	30°00'.0	0	2251.6	49.54	.....	.....	0	1300.0	60.76	.....	.....
$\frac{1}{4}$	2581.9	29 50'.7	561.4	2239.3	48.57	0.97	.....	323.1	1284.9	60.03	0.73	.....
$\frac{1}{2}$	2564.0	29 41'.4	1119.7	2227.3	47.63	0.94	0.03	642.5	1270.0	59.32	0.71	0.02
$\frac{3}{4}$	2546.4	29 32'.2	1675.1	2215.5	46.73	0.90	0.04	958.0	1255.3	58.64	0.68	0.03
1	(2529.2)	(29 22'.6)	2227.5	(2203.9)	(45.87)	(0.86)	(0.04)	(1270.0)	(1240.7)	(57.99)	(0.65)	(0.03)

First Prediction

$$\begin{aligned} x_1'' &= 46.73 - 0.86 = 45.87 \\ y_1'' &= 58.64 - 0.65 = 57.99 \\ x_m'' &= \frac{1}{2}(46.73 + 45.87) = 46.30 \text{ f.s.s.} \\ y_m'' &= \frac{1}{2}(58.64 + 57.99) = 58.32 \text{ f.s.s.} \\ x_1' &= 2215.5 - \frac{1}{4} \times 46.30 = 2203.9 \text{ f.s.} \\ y_1' &= 1255.3 - \frac{1}{4} \times 58.30 = 1240.7 \text{ f.s.} \\ y_m' &= \frac{1}{2}(1255.3 + 1240.7) = 1248.0 \text{ f.s.} \\ \Delta y &= \frac{1}{4} \times 1248.0 = 312.0 \text{ ft.} \\ y_1 &= 958.0 + 312.0 = 1270.0 \text{ ft.} \end{aligned}$$

$$\begin{aligned} x' &= 2203.9, \log 3.34319, 2\log 6.68638, (x')^2 = 4,857,100 \\ y' &= 1240.7, \log 3.09367, 2\log 6.18734, (y')^2 = 1,539,400 \\ v_1^2 & \dots \dots \dots 6,396,500 \\ 6,396,500, \log 6.80595, \frac{1}{2}\log 3.40298, v_1 &= 2529.2 \text{ f.s.} \end{aligned}$$

$$\begin{aligned} x' &= 2203.9 \dots \dots \dots \log 3.34319 \\ y' &= 1240.7 \dots \dots \dots \dots \dots \log 3.09367 \\ v_1 &= 2529.2 \dots \log 3.40298 \dots \text{colog } 6.59702 - 10 \dots \text{colog } 6.59702 - 10 \\ \theta_1 &= 29^\circ 22' 38'' \dots \dots \dots 1 \cos 9.94021 - 10 \dots \dots \dots 1 \sin 9.69069 - 10 \end{aligned}$$

Second Prediction

From  $H$ -table, with 1270 feet,  $\log H_v = 9.98258 - 10$

From  $G$ -table, with 2529.2 f.s.,  $G_v = 736.4$

$$\begin{aligned} G_v &= 736.4 \dots \dots \dots \log 2.86711 \\ H_v & \dots \dots \dots \log 9.98258 - 10 \\ C & \dots \dots \dots \text{colog } 8.87135 - 10 \end{aligned}$$

$$\begin{aligned} E_1 & \dots \dots \dots \log 1.72104 \dots \dots \dots \log 1.72104 \\ \theta_1 &= 29^\circ 22' 38'' \dots \dots \dots 1 \cos 9.94023 - 10 \dots \dots \dots 1 \sin 9.69069 - 10 \end{aligned}$$

$$\begin{aligned} x_1'' &= 45.84 \text{ f.s.s.} \dots \dots \dots \log 1.66127 \\ \hline 25.81 & \dots \dots \dots \log 1.41173 \\ 32.16 & \\ \hline y_1'' &= 57.97 \text{ f.s.s.} \end{aligned}$$

The difference between the first and second predictions is .03 f.s.s. in  $x''$ ; the change caused in  $x'$  by this amount would be less than .01 f.s., hence the first predictions were correct for all elements except  $x''$ , and for the latter the second prediction is correct. Finally, we have,

$$\begin{aligned} x_m' &= \frac{1}{2}(2215.5 + 2203.9) = 2209.7 \text{ f.s.} \\ \Delta x &= \frac{1}{4} \times 2209.7 = 552.4 \text{ feet} \\ x_1 &= 1675.1 + 552.4 = 2227.5 \text{ feet} \end{aligned}$$

620. We have now established the position of the projectile at four points separated by equal increments of time. With the information we have gained of the rate of change of the various elements, it is possible to increase the interval, and it will be most convenient to *double* it. In order to use the information already obtained we must combine it into two equal intervals, which means in this case two  $\frac{1}{2}$ -second intervals. This is done simply by tabulating the data for the points at times 0,  $\frac{1}{2}$  and 1, as already found. The tabulation follows:

Doubling  
the size of  
the interval

FIFTH INTERVAL

The work for the  $\frac{1}{2}$ -second interval is identical with that for the  $\frac{1}{4}$ -second intervals except that we must, of course, compute changes for  $\frac{1}{2}$  second instead of  $\frac{1}{4}$  second.

$t$	$v$	$\theta$	$x$	$x'$	$x''$	$\Delta_1$	$\Delta_2$	$y$	$y'$	$y''$	$\Delta_1$	$\Delta_2$
0	2600.0	30° 00'.0	0	2251.6	49.54	.....		0	1300.0	60.76	.....	.....
$\frac{1}{2}$	2564.0	29 41'.4	1119.7	2227.3	47.63	1.91	.....	642.5	1270.0	59.32	1.44	.....
1	2529.2	29 22'.6	2227.5	2203.9	45.84	1.79	0.12	1270.0	1240.7	57.97	1.35	0.09
$1\frac{1}{2}$	(2495.5)	(29 03'.7)	3323.8	(2181.4)	(44.17)	(1.67)	(0.12)	(1883.2)	(1212.0)	(56.71)	(1.26)	(0.09)

First Prediction

$$\begin{aligned} x_{3/2}'' &= 45.84 - 1.67 = 44.17 \\ y_{3/2}'' &= 57.97 - 1.26 = 56.71 \\ x_m'' &= \frac{1}{2}(45.84 + 44.17) = 45.00 \text{ f.s.s} \\ y_m'' &= \frac{1}{2}(57.97 + 56.71) = 57.34 \text{ f.s.s} \\ x_{3/2}' &= 2203.9 - \frac{1}{2} \times 45.00 = 2181.4 \text{ f.s.} \\ y_{3/2}' &= 1240.7 - \frac{1}{2} \times 57.34 = 1212.0 \text{ f.s.} \\ y_m' &= \frac{1}{2}(1240.7 + 1212.0) = 1226.4 \text{ f.s.} \\ \Delta y &= \frac{1}{2} \times 1226.4 = 613.2 \text{ ft.} \\ y_{3/2} &= 1270.0 + 613.2 = 1883.2 \text{ ft.} \end{aligned}$$

$$\begin{aligned} x' &= 2181.4, \log 3.33874, 2\log 6.67748, (x')^2 = 4,758,600 \\ y' &= 1212.0, \log 3.08350, 2\log 3.16700, (y')^2 = 1,468,900 \\ v_{3/2}^2 &= \dots\dots\dots 6,227,500 \\ 6,227,500, \log 6.79432, \frac{1}{2}\log 3.39716, v_{3/2} &= 2495.5 \text{ f.s.} \end{aligned}$$

$$\begin{aligned} x' &= 2181.4 \dots\dots\dots \log 3.33874 \\ y' &= 1212.0 \dots\dots\dots \dots\dots \log 3.08350 \\ v_{3/2} &= 2495.5 \dots \log 3.39716 \dots \underline{\text{colog } 6.60284 - 10} \dots \underline{\text{colog } 6.60284 - 10} \\ \theta_{3/2} &= 29^\circ 03' 23'' \dots\dots\dots \text{lcos } 9.94158 - 10 \dots \text{lsin } 9.68634 - 10 \end{aligned}$$

Second Prediction

From  $H$ -table, with 1883 feet,  $\log H_y = 9.97417 - 10$   
 From  $G$ -table, with 2495.5 f.s.,  $G_v = 721.2$

$$\begin{aligned} G_v &= 721.2 \dots\dots\dots \log 2.85806 \\ H_y &\dots\dots\dots \log 9.97417 - 10 \\ C &\dots\dots\dots \underline{\text{colog } 8.87135 - 10} \\ E_{3/2} &\dots\dots\dots \log 1.70358 \dots\dots\dots \log 1.70358 \\ \theta_{3/2} &= 29^\circ 03' 23'' \dots\dots\dots \text{lcos } 9.94158 - 10 \dots \text{lsin } 9.68634 - 10 \\ x_{3/2}'' &= 44.17 \text{ f.s.s.} \dots\dots\dots \log 1.64516 \\ \hline &24.54 \dots\dots\dots \log 1.38992 \\ &32.16 \\ \hline y_{3/2}'' &= 56.70 \text{ f.s.s.} \end{aligned}$$

Since no further changes will be found, due to the close agreement between the two predictions of  $x''$  and  $y''$ , we complete the interval by finding  $x$ .

$$\begin{aligned}x_m' &= \frac{1}{2}(2203.9 + 2181.4) = \underline{2192.6 \text{ f.s.}} \\ \Delta x &= \frac{1}{2} \times 2192.6 = \underline{1096.3 \text{ feet}} \\ x_{2/2} &= 2227.5 + 1096.3 = \underline{3323.8 \text{ feet}}\end{aligned}$$

621. By continuing the process outlined above the entire trajectory may be solved. The following additional features are encountered as the solution progresses.

Finding  
the summit

(a) The summit of the trajectory is defined by the point where  $y' = 0$ ,  $x' = v$ , and  $\theta = 0$ ; the value of  $y$  at this point is the maximum ordinate  $y_s$ . It will usually be necessary to interpolate within one of the chosen intervals in order to determine exactly where this occurs. Beyond the summit,  $y'$ ,  $\Delta y$ , and  $\theta$  all become negative, and  $y'$  increases,  $y$  decreases, and  $\theta$  increases. With  $x'$  continuing to decrease, while  $y'$  increases, a point is reached in the descending branch where the increases in  $y'$  just balance the decreases in  $x'$  (i.e., in formula (606)); at this point  $v$  has its least value and beyond it  $v$  increases.

Finding  
the point  
of fall

(b) The point of fall is defined by the point where  $y$  again becomes zero. Interpolation within the last interval will probably be necessary to determine exactly where this occurs. The values of  $t$ ,  $v$ ,  $\theta$ , and  $x$  corresponding to  $y = 0$  are, respectively,  $T$ ,  $v_\omega$ ,  $\omega$ , and  $X$  for the trajectory.

622. The portion of the illustrative problem actually solved above demonstrates adequately the principles involved in the solution of an entire trajectory. In order not to obscure the essential features of the method by details of labor-saving technique, the process illustrated has been left in a very elementary stage

Devices for  
shortening the  
computational  
work

insofar as computational features are concerned. Considering that trajectories involving times of flight up to about 100 seconds occur in practice, it is obvious that the shortening of the process of solution is an important consideration. The principal features

leading to a shortening of the process are as follows:

(a) The integrations are carried forward by the use of difference formulas,\* which lead more directly to the successive points.

(b) The size of the interval is increased as the work progresses. This is generally accomplished by doubling the size of the interval successively as soon as second differences for the increased size of interval become available (as illustrated in article 620). This successive doubling of the interval is generally carried on until the interval is 2 seconds; sometimes 4-second intervals are used, although the latter are rather large for trajectory computations. The maximum size of the interval is generally governed by the consideration that the required degree of accuracy in the predictions be obtained by the use of no higher than second differences.

\* The use of difference formulas is covered in most treatises on the solution of differential equations. A concise exposition is given on pp. 115 and 159 of Marks' *Mechanical Engineers' Handbook*, Second Edition. Further demonstrations of the solution of trajectories by numerical integration are given in the following references: Chapter III, *New Methods in Exterior Ballistics*, F. R. Moulton; *The Method of Numerical Integration in Exterior Ballistics*, Dunham Jackson (U. S. War Department Document No. 984, 1919); *A Course in Exterior Ballistics*, R. S. Hoar (U. S. War Department Document No. 1051, 1920).

623. Since the gravitational constant  $g$  varies according to latitude and height above the earth's surface, it is clear that both of these considerations enter into the solution of a trajectory by affecting the value of  $g$  used therein. We will examine briefly the complications that are introduced into our problem by these variations in  $g$ , and what degree of error results from neglecting them.

It is true, strictly speaking, that a trajectory solution which is correct for one locality cannot also be correct for another having a different latitude. The sea-level value of  $g$  varies from 32.087 f.s.s. at the equator to 32.257 f.s.s. at the poles.\* The value  $g = 32.16$  f.s.s. (corresponding to sea level at latitude about  $41^\circ$ ) is used by the U. S. Navy for trajectory computations involving  $g$  directly; at the U. S. Naval Proving Ground,  $g = 32.155$  f.s.s. is used in experimental work. The U. S. Army uses  $g = 9.80$  m.s.s. = 32.152 f.s.s.; the French generally use  $g = 9.81$  m.s.s. = 32.185 f.s.s., but for close work sometimes use  $g = 9.8085$  m.s.s. = 32.180 f.s.s. (all of these are values at sea level). But when a value of  $i$  is determined from experimental firing, as will be outlined in the next article, the value of  $i$  thus obtained necessarily accounts for any difference between the values of  $g$  as assumed in the computation of the ballistic tables used in this determination, and as actually existing at the locality of the experimental firing. The range table computed according to this value of  $i$  and with the same ballistic tables is thus adjusted, in effect, for the value of  $g$  actually existing at the locality at which the experimental determination of  $i$  was carried out, regardless of what may have been the value of  $g$  assumed for the ballistic tables. A correction might now be deduced for adjusting the range-table values of elements of the trajectory for the change in  $g$  with latitude, but this is not considered necessary, in view of the small changes involved. Such changes in  $g$  as occur within the extreme limits of latitude involved in practice are of small importance, especially in comparison with such errors as may exist in the retardation function itself.

The situation with respect to the variation of  $g$  with altitude is much the same. The differences between values of  $g$  at the surface and at the greatest ordinates occurring in artillery practice remain small enough to be inconsiderable in comparison with the probable inaccuracies of the retardation function. A further inaccuracy results from considering  $g$  to act always parallel to the original  $Y$ -axis, whereas  $g$  actually acts always directly toward the center of the earth, i.e., vertically with respect to the curved surface of the earth rather than with respect to the flat plane tangent to the earth at the origin of the trajectory. But the change in direction of  $g$  with respect to the assumed flat plane is so small (being only about  $1^\circ$  in 120,000 yards) that the error resulting from neglecting it is very small. Moreover, both of these slight inaccuracies also are eventually accounted for in the value of  $i$ .

624. Having arrived at the conclusion, in article 417, that the evaluation of the coefficient of form ultimately must depend upon measurements applied to an entire trajectory, it is a natural query at this point how we are to perform the

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\* Helmert's formula is generally accepted in ballistics. Omitting terms pertaining to height above sea level and to local abnormalities in density of the earth's structure, the formula is  $g = (32.172 - 0.0853 \cos 2l)$  f.s.s.,  $l$  being the latitude. The term for change of  $g$  with altitude gives this change to be  $(-)$  .003 f.s.s. per 1000 feet above sea level. Ref. Chapter II, *New Methods in Exterior Ballistics*, Moulton; also pp. VIII and IX, *Exterior Ballistic Tables Based on Numerical Integration*, Vol. I, 1924 (U. S. Army); also *T.S. No. 148*, April, 1921, (U. S. Army).

above solution, which requires knowledge of the value of  $i$ , if the value of  $i$  in turn depends upon the solution itself. The situation is such that, having actually fired a gun under a given set of conditions (i.e., with  $\phi$ ,  $V$ ,  $\delta$ ,  $w$ , and  $d$  all carefully measured), we measure the resulting range  $X$ , and the problem then is to find the value of  $i$  which, when combined with the identical  $\phi$ ,  $V$ ,  $\delta$ ,  $w$ , and  $d$  used in the firing, will reproduce the identical  $X$  actually measured. One way of accomplishing this would be to proceed by making a solution with an estimated value of  $i$  and, by comparing the resulting computed range with the required value, making a better estimate of  $i$  and a further solution, and so on—in short, by a process of successive approximations. Even with absolutely no prior knowledge as to the value of  $i$  we might thus eventually arrive at its value, although at the expense of great labor.

The equivalent of the above is indeed what is resorted to, although after a certain amount of preliminary work has been done and recorded in advantageous form, the remaining process for any given situation is reduced to exceedingly simple terms. Solutions are made for a great many trajectories, embracing all values of  $\phi$ ,  $V$ , and  $C$  that are likely to occur in practice. The computed values of the terminal elements  $X$ ,  $\omega$ ,  $T$ ,  $v_\omega$ , and of  $y_s$ , are then tabulated against the arguments  $\phi$ ,  $V$ , and  $C$  in such manner that the values of these elements may be found for any combination of values of the arguments. Now if a gun is fired at a known angle of departure  $\phi$  and the initial velocity  $V$  of the shot is measured, and if the resulting range  $X$  also is measured, then it is a simple matter to find in our table what value of  $C$  corresponds to the given values of  $\phi$ ,  $V$ , and  $X$ . Having thus determined the value of  $C$ , and having also measured  $\delta$ ,  $w$ , and  $d$ , it then remains but to find the value of  $i$  which, in combination with the given  $\delta$ ,  $w$ , and  $d$ , satisfies the given value of  $C$  (i.e., we solve for  $i$  by means of (406) transposed to the

form  $i = \frac{w}{\delta C d^2}$ ). Further details of this process will be dealt with in the next

chapter, and the principal features of the proving-ground phases involved therein will be discussed in Chapter 8.

625. It would appear, without giving due thought to the matter, that if it is necessary to determine  $X$  in the first place in order to determine a value of  $i$ , it should be unnecessary to do anything further at all than to record the value of  $X$  for future reference—for after all  $i$  serves no better purpose than to permit the already known value of  $X$  to be reproduced by computation. This would be true if by experimental firing alone we could measure, not only  $X$  but all other required elements ( $\omega$ ,  $T$ ,  $v_\omega$ ,  $y_s$ ), for all probable combinations of  $\phi$ ,  $V$ ,  $\delta$ ,  $w$ , and  $d$ . This is altogether impracticable, not only because of the enormous amount of firing\* that it would involve but also because it is practically impossible to measure such elements as  $\omega$ ,  $v_\omega$ , and  $y_s$ . Moreover, it is necessary also to know the values of elements at various points of the trajectory other than the point of fall, not only for antiaircraft work but also in order that corrections may be deduced for influences that vary in amount for different portions of the trajectory, as for example winds which vary at different altitudes (as they usually do).

\* Such firing takes a great deal of time and is very expensive, since it involves not only the direct cost of guns and ammunition but also very considerable overhead costs for operating personnel, triangulation parties, patrol vessels, etc. The cost of only a few experimental shots from a major-caliber gun is sufficient to pay the annual wage of an expert computer.

It is true, strictly speaking, that a different value of  $i$  may be required for each different combination of  $\phi$ ,  $V$ ,  $\delta$ ,  $w$ , and  $d$ , but it is also true that  $i$  varies slowly enough to permit interpolation between values determined experimentally for fairly widely separated combinations of the above elements. From experience it is found that experimental firing at relatively few angles of departure is sufficient for the construction of an entire range table for a given  $V$  and  $C$ , and for ordinary variations in the latter (such as variations in  $V$  due to erosion and powder temperature, and in  $C$  due to non-standard  $\delta$ ).

626. All that has been said with respect to the determination of the value of  $i$  for the type of solution illustrated in this chapter, applies equally well to any other type of solution of a trajectory. In Siacci's Method also, it is necessary to make what amounts to a solution backwards from a measured  $X$  and thus to find the value of  $i$  which satisfies the known value of  $X$ . And in Siacci's Method the process of making this backward solution is similar in nature to that outlined above for the numerical integration method, although it is somewhat less direct.\* It should be apparent, then, that Siacci's Method or any other method may be forced, by determination of the appropriate  $i$ , to reproduce the actually measured  $X$ , just as the numerical integration was forced so to do. Also, it should be ap-

The dependence of  $i$  upon the method of solution

parent that the value of  $i$  determined by a process of this character will, for the same set of conditions, vary according to the method of solution in which it is to be used. For we now find that, in addition to having to account for the many imperfectly known influences already discussed in articles 413-417,  $i$  has to account also for the differences in the degrees of approximation of the various methods of solution.

The fact that, by determination of the appropriate value of  $i$ , Siacci's Method can be made to produce just as correct a value of the range as can be gotten by the more laborious method of numerical integration, does not alter the situation that the latter affords an appreciably more accurate determination for points other than the point of fall. Siacci's Method holds a very decided advantage over the numerical integration method for finding terminal elements of trajectories of limited curvature; it does not, however, adapt itself satisfactorily to accurate determination of elements at intermediate points, for reasons already outlined in article 514.

627. It must be borne in mind, however, that the advantages as to accuracy that we have ascribed to the numerical integration method pertain entirely to the process of solution and do not overcome the difficulties arising from imperfect knowledge of physical values involved in the solution, except insofar as they afford a better separation of the effects of the latter from the effects of inaccuracies in the process of solution itself.

Limitations of accuracy of the numerical integration method

The numerical integration process may be said to afford accuracy in the solution for any point of a trajectory up to the limits of accuracy within which the physical data pertaining to that point are known. In the particular case of the terminal point, the sum total effect of inaccuracies of all

kinds, including those pertaining to physical data, can be wiped out by determining  $i$  as has already been described above. For intermediate points we depend on the assumption that  $i$  remains the same for all points in the trajectory, although this is not necessarily true.

By using the numerical integration process of solution we afford all intermediate points the same treatment as we do the point of fall, and thus eliminate

\* See Appendix A.

changes in  $i$  which are incident to varying degrees of approximation for various portions of the same trajectory (such as are inherent in Siacci's Method). But in so doing we do not eliminate the possibility that  $i$  may vary along the trajectory due to physical causes. Thus the  $i$  determined for the point of fall includes, among other things, the effect of obliquity of the projectile as an average for the whole trajectory (art. 417); the average obliquity for the portion of the trajectory up to some intermediate point may, however, differ from the average for the whole trajectory, and the  $i$  for the intermediate point accordingly may differ from the  $i$  for the whole trajectory. More particularly it is probable that the assumed retardation function (such as the  $G$ -function) does not fit the given case exactly. The  $i$  determined for the point of fall wipes out the accumulated error, for the whole trajectory, due to the inaccuracy of the assumed retardation function. Any difference between the error accumulated up to an intermediate point and that accumulated up to the point of fall requires an adjustment of  $i$  for the intermediate point.

In the absence of explicit knowledge of each and every particle of physical data pertaining to each and every point in the trajectory, it becomes necessary to determine  $i$  for each point in a manner similar to that already described for the point of fall. Since the latter is generally impracticable, the accuracy of solutions for intermediate points is always limited by our knowledge of the necessary physical data.

628. At the Aberdeen Proving Ground recently, in the firing of .30 caliber and .50 caliber machine guns for which trajectories had been computed according to the  $G$ -function and with values of  $i$  that satisfied observed ranges, it was observed that the computed trajectories failed materially to agree at intermediate points with the observed trajectories (as indicated by tracers). Dr. Hedrick of the Aberdeen Proving Ground then invented a purely empirical method of varying  $i$  along the trajectory to secure a more satisfactory agreement. He found, in this case, that it was necessary to change  $i$  from point to point until a certain portion of the trajectory had been computed, and that thereafter a constant  $i$  could be used for the remainder of the trajectory. The difficulty in this case was ascribed to failure of the  $G$ -function to apply with sufficient accuracy to the projectile in use, and particularly so at certain velocities.

The U. S. Army has in recent years engaged in extensive research work for the purpose of establishing better retardation functions, and these efforts have materialized in the direction of establishing a number of such functions, a different one for each of several types of projectile. As better retardation data become available the advantages of accuracy inherent in the numerical integration method of solution will be gained more fully.

629. The use of ballistic tables has already been touched upon in article 624. Such tables not only constitute the practical basis for determining the value of  $i$ , but they are a great convenience for the computation of range tables. Nearly all methods of solving the trajectory that have been developed in the past have given rise to ballistic tables in which the results of the more tedious operations have been tabulated against convenient arguments. The nature of the ballistic tables pertaining to Siacci's Method has already been touched upon in article 511. The ballistic tables of the numerical integration method in its modern form afford a more direct approach to the ultimate, practical application to the construction of range tables than has been the case with ballistic tables of the various approximation methods (such as Siacci's). The greater labor involved in the basic computations according to the numerical integration method is therefore com-

pensated for in the simpler process remaining for its practical application to the preparation of range tables.

In addition to the various labor-saving devices that may be developed in the computational process itself, it is noteworthy that the computation of one long trajectory by numerical integration may be made to yield data for many shorter trajectories included within the limits of the long trajectory. This will be apparent from a study of Figure 13. Let us assume that the trajectory  $OSH$  of Figure 13 represents the trajectory for the conditions stated in article 611, and that the values of  $t$ ,  $v$ ,  $\theta$ ,  $x$ , and  $y$  have been recorded for the various intervals used in the solution. We can then find the values of all the elements at any point (by interpolation, if neces-

Computation  
of ballistic  
tables based  
on numerical  
integration

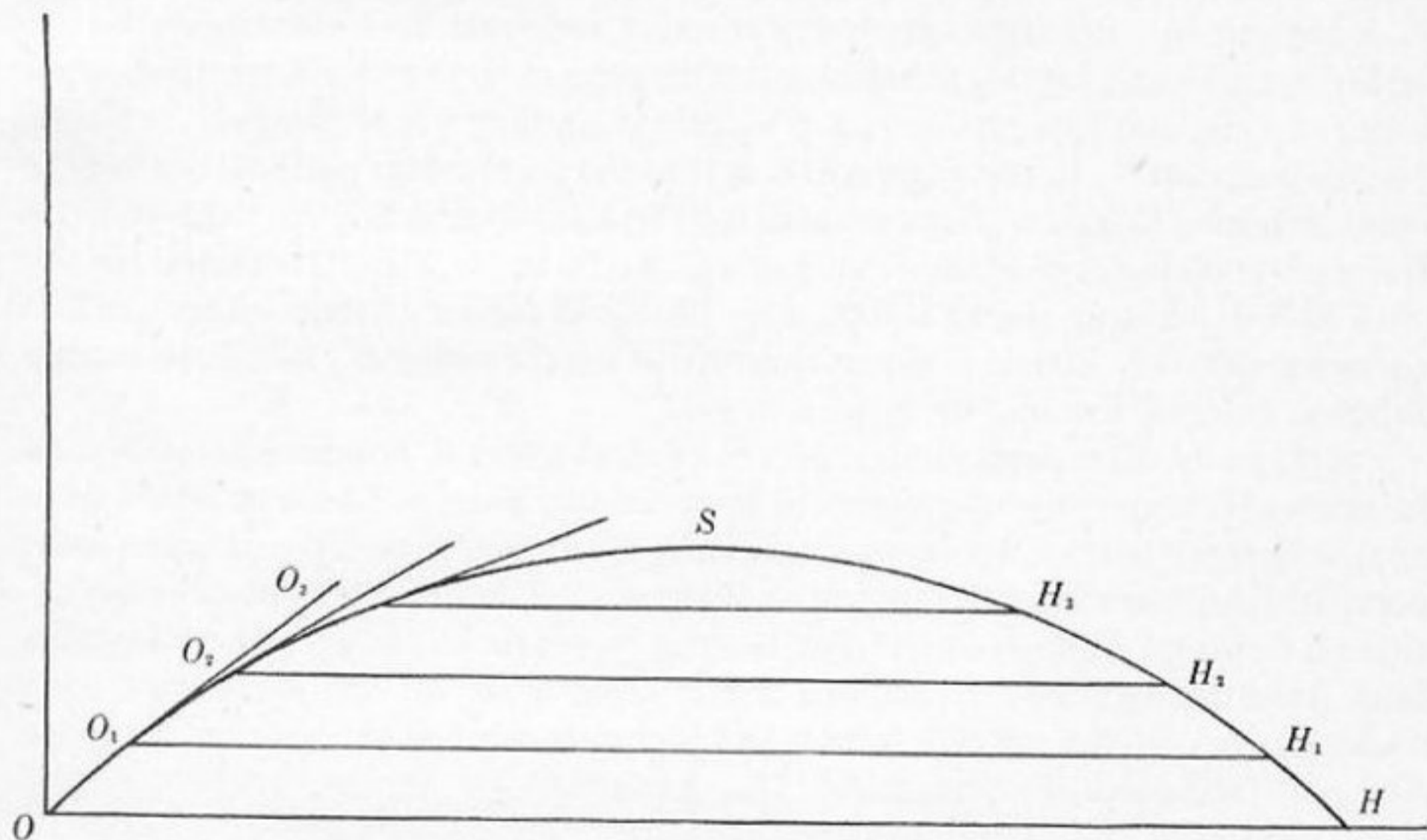


FIGURE 13

sary), as for example at the point  $O_1$ , which we will assume to be the point where  $\theta = 25^\circ$ . For this point  $O_1$ , where  $\theta = 25^\circ$ , we can then define  $t$ ,  $v$ ,  $x$ , and  $y$ . We also can find in the descending branch the point  $H_1$  which lies in the same horizontal plane as  $O_1$ ; this will be the point where  $y$  has the same value that  $y$  has at the point  $O_1$ . Also, we can find  $t$ ,  $v$ ,  $\theta$ , and  $x$  for the point  $H_1$ . We now have sufficient data for the trajectory  $O_1SH_1$ , which is the trajectory for the conditions  $\phi = \theta = 25^\circ$ ,  $V = v$  (at  $O_1$ ),  $v_\omega = v$  (at  $H_1$ ), and  $\omega = \theta$  (at  $H_1$ ). Also,  $X$  for the trajectory  $O_1SH_1$  equals  $x$  (at  $H_1$ ) minus  $x$  (at  $O_1$ );  $T$  equals  $t$  (at  $H_1$ ) minus  $t$  (at  $O_1$ ); and  $y_s$  equals  $y_s$  for the trajectory  $OSH$  minus  $y$  (at  $O_1$  and  $H_1$ ). Numerous other trajectories, such as  $O_2SH_2$ ,  $O_3SH_3$ , etc., can be determined similarly.

630. Since atmospheric density varies with altitude we must, however, adjust the value of  $C$  to the zero level of each trajectory cut from the principal trajectory. This is done quite simply by letting  $C$  for the zero level of any included trajectory

equal  $\frac{C}{H_v}$  of the principal trajectory for the level of the latter which corresponds

to the zero level of the former. For example, let us assume that at  $O_1$  the altitude is  $y = 8000$  feet, for which  $\log H_v = 9.89024 - 10$ , and that  $\log C = 1.12865$ . It is

immaterial whether we use, at point  $O_1$ ,  $E = \frac{G_v \times H_v}{C}$  with  $\log H_v$  and  $\log C$  having

the values stated above, or  $E = \frac{G_v \times H_v}{C}$  with  $H_v$  equal to unity and  $\log C$

having the value  $(1.12865) - (9.89024 - 10) = 1.23841$ . That is, the value  $\log C = 1.23841$  corresponds to zero altitude for the trajectory  $O_1SH_1$ , and in tabulating the elements for this trajectory we must tabulate them not only against the  $\phi$  and  $V$  found as already noted above, but also against the  $\log C$  here found.

This process is valid because  $H_v$  is an exponential function, and we can verify this by a numerical example as follows. It should be possible to compute a point lying beyond  $O_1$  with identical results whether we work from the origin and use  $\log C = 1.12865$  and  $\log H_v$  as measured with respect to the origin, or whether we work from the point  $O_1$  and use  $\log C = 1.23841$  and  $\log H_v$  as measured with respect to the point  $O_1$ . Let us suppose, then, that the point where  $y = 9000$  feet is to be found. Working from the origin we have  $\log C = 1.12865$  and  $\log H_v = 9.87652 - 10$  (using  $y = 9000$  feet); working from point  $O_1$  we have  $\log C = 1.23841$  and  $\log H_v = 9.98628 - 10$  (using  $y = 1000$  feet, i.e., 1000 feet higher than 8000 feet, which is the level at  $O_1$ ). Either of these two sets of  $\log C$  and  $\log H_v$  results in exactly the same value of  $E$  when substituted in (601).

It appears, therefore, that each set of  $\phi$ ,  $V$ , and  $C$  assumed for the computation of a trajectory by numerical integration yields, in addition to the principal trajectory defined by these arguments, a number of additional trajectories defined by an assortment of sets of  $\phi$ ,  $V$ , and  $C$ . The arrangement of these additional solutions against convenient tabular intervals of the arguments involves much painstaking interpolation, but nevertheless it affords an economy of labor in comparison to the very tedious step-by-step numerical integration for all of these additional sets of arguments.\*

### EXERCISES

1. Using the same data and process as stated for the example given in article 604, compute the distances covered by the projectile in the intervals while its velocity was being reduced, (a) from 1800 f.s. to 1790 f.s., and (b) from 1790 f.s. to 1780 f.s.

*Answer* (a) 42.0 feet (b) 42.2 feet.

2. Using the same data and process as stated for the example given in article 604, compute the distance covered by the projectile while its velocity was being reduced from 1820 f.s. to 1810 f.s., making the computation in five 2 f.s. intervals.

*Answer* 41.7 feet.

3. Using the same data and process as stated for the example in article 608, compute the distances covered by the projectile during the following intervals, and remaining velocities at the ends of these intervals: (a) 0.2 sec. to 0.3 sec. (b) 0.3 sec. to 0.4 sec. (c) 0.4 sec. to 0.5 sec.

*Answer* (a) 169.8 feet (b) 166.0 feet (c) 162.4 feet  
1678.8 f.s. 1641.6 f.s. 1605.9 f.s.

\* It affords, also, a means of tabulating not only terminal points but many intermediate points, without increasing the number of arguments. Volume I of the U. S. Army's *Exterior Ballistic Tables Based on Numerical Integration* is based on this principle. Further details may be found in the introductory pages of these tables.

4. Using the same data and process as stated for the example in article 608, compute the distance covered by the projectile in the first 0.1 sec., making the computation in two intervals of 0.05 sec. each.

*Answer* 177.9 feet.

5. Using the same data and process as stated for the example in article 608, compute the distance covered by the projectile in the first 0.5 sec., and remaining velocity at the end of that time, making the computation in one interval. (Compare with results given in last paragraph of article 608.)

*Answer* 851.3 feet; 1605.3 f.s.

6. Continue the problem stated in article 611 to the point  $t=6$  seconds, making the computations for one  $\frac{1}{2}$ -second interval.

## ANSWERS

$t$	$v$	$\theta$	$x$	$x'$	$x''$	$y$	$y'$	$y''$
2	2463.2	28°44'6	4409.1	2159.7	42.59	2482.2	1183.9	55.52

TABLE I

The Gâvre Retardation Function  $G_v$ 

This table is entered with the velocity  $v$ , in foot-seconds, and the value of  $G_v$  is found from the body of the table. The value of  $G_v$ , thus found is the retardation in foot-seconds per second, encountered by a projectile of unit ballistic coefficient in atmosphere of standard surface density, at the given velocity. The derivation and use of this table are explained in articles 410-411, 428-429, and 604-621, *Exterior Ballistics, 1935*.

V	00	10	20	30	40	50	60	70	80	90	V	00	10	20	30	40	50	60	70	80	90
0	0.00	0.00	0.02	0.05	0.09	0.14	0.20	0.26	0.33	0.41	3000	973.0	978.5	984.0	989.5	995.0	1000.6	1006.3	1012.1	1017.9	1023.7
100	0.50	0.60	0.71	0.83	0.96	1.10	1.25	1.41	1.58	1.76	3100	1029.5	1035.2	1041.0	1046.8	1052.5	1058.3	1064.1	1070.0	1075.8	1081.6
200	1.95	2.14	2.33	2.53	2.74	2.96	3.19	3.43	3.67	3.91	3200	1087.5	1093.5	1099.6	1105.8	1112.1	1118.4	1124.6	1130.7	1136.9	1143.1
300	4.16	4.42	4.68	4.95	5.23	5.51	5.80	6.10	6.40	6.71	3300	1149.3	1155.6	1162.0	1168.3	1174.5	1180.8	1187.2	1193.7	1200.1	1206.6
400	7.03	7.36	7.69	8.02	8.36	8.70	9.05	9.42	9.80	10.19	3400	1213.2	1219.8	1226.4	1233.0	1239.6	1246.2	1252.7	1259.3	1266.0	1272.8
500	10.58	10.97	11.37	11.78	12.19	12.62	13.05	13.49	13.95	14.43	3500	1279.6	1286.5	1293.5	1300.5	1307.5	1314.4	1321.3	1328.3	1335.2	1342.2
600	14.92	15.42	15.93	16.44	16.97	17.52	18.09	18.69	19.31	19.95	3600	1349.3	1356.4	1363.5	1370.6	1377.7	1384.8	1392.0	1399.2	1406.5	1413.8
700	20.60	21.28	21.98	22.71	23.48	24.28	25.11	25.97	26.87	27.82	3700	1421.2	1428.5	1435.8	1443.2	1450.7	1458.2	1465.7	1473.2	1480.6	1488.0
800	28.82	29.87	31.02	32.20	33.43	34.73	36.11	37.58	39.13	40.77	3800	1495.5	1503.0	1510.5	1518.0	1525.6	1533.2	1540.9	1548.7	1556.4	1564.0
900	42.61	44.47	46.46	48.58	50.85	53.28	55.87	58.63	61.57	64.70	3900	1571.6	1579.2	1586.8	1594.5	1602.2	1610.0	1617.8	1625.7	1633.7	1641.7
1000	68.0	71.7	75.6	79.6	83.8	88.2	92.8	97.5	102.4	107.5	4000	1649.6	1657.6	1665.7	1673.7	1681.8	1689.8	1698.0	1706.1	1714.3	1722.4
1100	112.7	118.0	123.4	128.8	134.2	139.6	145.0	150.4	155.8	161.1	4100	1730.6	1738.9	1747.2	1755.4	1763.7	1772.0	1780.4	1788.8	1797.1	1805.5
1200	166.4	171.6	176.8	181.9	186.9	191.8	196.7	201.6	206.5	211.4	4200	1813.9	1822.4	1830.9	1839.3	1847.8	1856.3	1864.9	1873.4	1882.0	1890.6
1300	216.2	220.9	225.5	230.1	234.7	239.3	243.8	248.3	252.8	257.2	4300	1899.2	1907.9	1916.6	1925.3	1934.0	1942.7	1951.5	1960.3	1969.2	1978.0
1400	261.6	266.1	270.5	274.9	279.3	283.6	287.8	292.1	296.3	300.6	4400	1986.8	1995.7	2004.7	2013.6	2022.6	2031.5	2040.5	2049.5	2058.6	2067.6
1500	304.9	309.2	313.5	317.7	321.8	326.0	330.2	334.5	338.8	343.0	4500	2076.6	2085.7	2094.8	2103.9	2113.0	2122.1	2131.3	2140.5	2149.7	2158.9
1600	347.1	351.2	355.3	359.5	363.6	367.7	371.8	376.0	380.1	384.2	4600	2168.1	2177.4	2186.7	2196.1	2205.4	2214.7	2224.2	2233.6	2243.1	2252.5
1700	388.3	392.4	396.4	400.5	404.7	408.8	412.8	416.8	420.9	425.0	4700	2262.0	2271.6	2281.2	2290.8	2300.4	2310.0	2319.7	2329.4	2339.2	2348.9
1800	429.2	433.3	437.4	441.5	445.5	449.5	453.5	457.6	461.6	465.6	4800	2358.6	2368.4	2378.3	2388.1	2398.0	2407.8	2417.7	2427.6	2437.6	2447.5
1900	469.7	473.7	477.8	481.8	485.9	490.1	494.2	498.3	502.4	506.5	4900	2457.4	2467.4	2477.4	2487.4	2497.4	2507.4	2517.5	2527.6	2537.7	2547.8
2000	510.5	514.5	518.6	522.7	526.8	530.9	535.0	539.2	543.3	547.4	5000	2557.9	2568.1	2578.3	2588.4	2598.6	2608.8	2619.1	2629.4	2639.7	2650.0
2100	551.6	555.7	559.9	564.1	568.2	572.4	576.5	580.6	584.7	588.8	5100	2660.3	2670.7	2681.2	2691.6	2702.1	2712.5	2723.1	2733.6	2744.2	2754.7
2200	593.0	597.2	601.4	605.6	609.7	613.9	618.2	622.4	626.6	630.8	5200	2765.3	2776.0	2786.6	2797.3	2807.9	2818.6	2829.4	2840.1	2850.9	2861.6
2300	635.1	639.5	643.8	648.1	652.5	656.9	661.2	665.5	669.8	674.1	5300	2872.4	2883.2	2894.0	2904.9	2915.7	2926.5	2937.4	2948.3	2959.2	2970.1
2400	678.5	682.9	687.4	691.8	696.2	700.6	705.1	709.7	714.2	718.7	5400	2981.0	2992.0	3003.0	3014.1	3025.1	3036.1	3047.2	3058.3	3069.5	3080.6
2500	723.2	727.7	732.2	736.8	741.5	746.1	750.6	755.2	759.9	764.6	5500	3091.7	3102.9	3114.2	3125.4	3136.7	3147.9	3159.3	3170.6	3182.0	3193.3
2600	769.3	774.0	778.7	783.5	788.2	793.0	797.8	802.7	807.6	812.5	5600	3204.7	3216.2	3227.7	3239.1	3250.6	3262.1	3273.7	3285.3	3297.0	3308.6
2700	817.4	822.2	827.1	832.0	836.9	841.8	846.8	851.9	857.0	862.0	5700	3320.2	3331.9	3343.6	3355.4	3367.1	3378.9	3390.6	3402.4	3414.2	3426.0
2800	867.1	872.1	877.1	882.2	887.3	892.5	897.8	903.0	908.3	913.6	5800	3437.8	3449.7	3461.6	3473.5	3485.4	3497.3	3509.3	3521.4	3533.4	3545.5
2900	918.9	924.2	929.5	934.8	940.2	945.6	951.0	956.5	962.0	967.5	5900	3557.5	3569.7	3581.9	3594.2	3606.4	3618.6	3631.0	3643.4	3655.7	3668.1
3000	973.0	978.5	984.0	989.5	995.0	1000.6	1006.3	1012.1	1017.9	1023.7	6000	3680.5									

The above table has been taken from *Eggert's Method*, as published in the *U. S. Naval Institute Proceedings*, Vol. 45, Nos. 191 and 200. It is based on tables originally published in France by the *Commission de Gâvre*. A table of the Gâvre function in metric units is given in *Exterior Ballistics Tables Based on Numerical Integration*, Vol. I, (Ordnance Department U. S. Army, 1924). See also note on page 28 *Exterior Ballistics, 1935*.

## TABLE II

### The Altitude-Density Function $H_y$

This table is entered with the altitude  $y$ , in feet, and the value of  $\log H_y$  is found from the body of the table.  $H_y$  expresses the ratio of air density at the altitude  $y$  feet to the density of air at the earth's surface at sea level, under the conditions of standard surface density (i.e.,  $\delta = 1$ ). The values of  $\log H_y$  were computed by means of the formula

$$\log H_y = -.00001372 y.$$

Particular attention is invited to the fact that this table gives  $\log H_y$ , and that the characteristic 9-10 is to be understood for all of the values given in the table. Although the table extends only to 10,000 feet for the argument  $y$ , it may be used for greater altitudes than 10,000 feet as follows. Divide the value of  $y$  by 10. Entering the table with  $y/10$ , find the corresponding value of  $\log H_y$  and move the decimal point of this value one place to the right. For example, to find  $\log H_y$  for  $y = 10,100$  feet, we enter with  $y = 1010$  feet and find in the body of the table the value .98614. Moving the decimal point one place to the right we then have  $\log H_y = 9.8614 - 10$ . It is to be noted that this operation gives also the characteristic of the logarithm. The function  $H_y$ , as defined above and as given by this table, is limited to altitudes within the troposphere (about 40,000 feet).

The derivation and use of this table are further explained in articles 420, 428-429, and 610-621, *Exterior Ballistics, 1935*.

y	0	10	20	30	40	50	60	70	80	90	y	0	10	20	30	40	50	60	70	80	90
000	.00000	.99986	.99973	.99959	.99945	.99931	.99918	.99904	.99890	.99877	5100	.93003	.92989	.92975	.92962	.92948	.92934	.92920	.92907	.92893	.92879
100	.99863	.99850	.99835	.99822	.99808	.99794	.99780	.99767	.99753	.99739	5200	.92866	.92852	.92838	.92824	.92811	.92797	.92783	.92770	.92756	.92742
200	.99726	.99712	.99698	.99684	.99671	.99657	.99643	.99630	.99616	.99602	5300	.92728	.92715	.92701	.92687	.92674	.92660	.92646	.92632	.92619	.92605
300	.99588	.99575	.99561	.99547	.99533	.99520	.99506	.99492	.99479	.99465	5400	.92591	.92577	.92564	.92550	.92536	.92523	.92509	.92495	.92481	.92468
400	.99451	.99437	.99424	.99410	.99396	.99383	.99369	.99355	.99341	.99328	5500	.92454	.92440	.92427	.92413	.92399	.92385	.92372	.92358	.92344	.92331
500	.99314	.99300	.99287	.99273	.99259	.99245	.99232	.99218	.99204	.99190	5600	.92317	.92303	.92289	.92276	.92262	.92248	.92234	.92221	.92207	.92193
600	.99177	.99163	.99149	.99136	.99122	.99108	.99094	.99081	.99067	.99053	5700	.92180	.92166	.92152	.92138	.92125	.92111	.92097	.92084	.92070	.92056
700	.99040	.99026	.99012	.98998	.98985	.98971	.98957	.98944	.98930	.98916	5800	.92042	.92029	.92015	.92001	.91988	.91974	.91960	.91946	.91933	.91919
800	.98902	.98889	.98875	.98861	.98848	.98834	.98820	.98806	.98793	.98779	5900	.91905	.91891	.91878	.91864	.91850	.91837	.91823	.91809	.91795	.91782
900	.98765	.98751	.98738	.98724	.98710	.98697	.98683	.98669	.98655	.98642	6000	.91768	.91754	.91741	.91727	.91713	.91699	.91686	.91672	.91658	.91645
1000	.98628	.98614	.98600	.98587	.98573	.98559	.98546	.98532	.98518	.98505											
1100	.98491	.98477	.98463	.98450	.98436	.98422	.98408	.98395	.98381	.98367	6100	.91631	.91617	.91603	.91590	.91576	.91562	.91548	.91535	.91521	.91507
1200	.98354	.98340	.98326	.98312	.98299	.98285	.98271	.98258	.98244	.98230	6200	.91494	.91480	.91466	.91452	.91439	.91425	.91411	.91398	.91384	.91370
1300	.98216	.98203	.98189	.98175	.98162	.98148	.98134	.98120	.98107	.98093	6300	.91356	.91343	.91329	.91315	.91302	.91288	.91274	.91260	.91247	.91233
1400	.98079	.98065	.98052	.98038	.98024	.98011	.97997	.97983	.97969	.97956	6400	.91219	.91205	.91192	.91178	.91164	.91151	.91137	.91123	.91109	.91096
1500	.97942	.97928	.97915	.97901	.97887	.97873	.97860	.97846	.97832	.97819	6500	.91082	.91068	.91055	.91041	.91027	.91013	.91000	.90986	.90972	.90959
1600	.97805	.97791	.97777	.97764	.97750	.97736	.97722	.97709	.97695	.97681	6600	.90945	.90931	.90917	.90904	.90890	.90876	.90862	.90849	.90835	.90821
1700	.97668	.97654	.97640	.97626	.97613	.97599	.97585	.97572	.97558	.97544	6700	.90808	.90794	.90780	.90766	.90753	.90739	.90725	.90712	.90698	.90684
1800	.97530	.97517	.97503	.97489	.97476	.97462	.97448	.97434	.97421	.97407	6800	.90670	.90657	.90643	.90629	.90616	.90602	.90588	.90574	.90561	.90547
1900	.97393	.97379	.97366	.97352	.97338	.97325	.97311	.97297	.97283	.97270	6900	.90533	.90519	.90506	.90492	.90478	.90465	.90451	.90437	.90423	.90410
2000	.97256	.97242	.97229	.97215	.97201	.97187	.97174	.97160	.97146	.97133	7000	.90396	.90382	.90369	.90355	.90341	.90327	.90314	.90300	.90286	.90273
2100	.97119	.97105	.97091	.97078	.97064	.97050	.97036	.97023	.97009	.96995	7100	.90258	.90244	.90227	.90214	.90201	.90188	.90175	.90162	.90149	.90135
2200	.96982	.96968	.96954	.96940	.96927	.96913	.96899	.96886	.96872	.96858	7200	.90121	.90108	.90094	.90080	.90067	.90054	.90049	.90026	.90012	.89998
2300	.96844	.96831	.96817	.96803	.96790	.96776	.96762	.96748	.96735	.96721	7300	.89984	.89971	.89957	.89943	.89930	.89916	.89902	.89888	.89875	.89861
2400	.96707	.96693	.96680	.96666	.96652	.96639	.96625	.96611	.96597	.96584	7400	.89847	.89833	.89820	.89806	.89792	.89779	.89765	.89751	.89737	.89724
2500	.96570	.96556	.96543	.96529	.96515	.96501	.96488	.96474	.96460	.96447	7500	.89710	.89696	.89683	.89669	.89655	.89641	.89628	.89614	.89600	.89587
2600	.96433	.96419	.96405	.96392	.96378	.96364	.96350	.96337	.96323	.96309	7600	.89573	.89559	.89545	.89532	.89518	.89504	.89490	.89477	.89463	.89449
2700	.96296	.96282	.96268	.96254	.96241	.96227	.96213	.96200	.96186	.96172	7700	.89436	.89422	.89408	.89394	.89381	.89367	.89353	.89340	.89326	.89312
2800	.96158	.96145	.96131	.96117	.96104	.96090	.96076	.96062	.96049	.96035	7800	.89298	.89285	.89271	.89257	.89244	.89230	.89216	.89202	.89189	.89175
2900	.96021	.96007	.95994	.95980	.95966	.95953	.95939	.95925	.95911	.95898	7900	.89161	.89147	.89134	.89120	.89106	.89093	.89079	.89065	.89051	.89038
3000	.95884	.95870	.95857	.95843	.95829	.95815	.95802	.95788	.95774	.95761	8000	.89024	.89010	.88997	.88983	.88969	.88955	.88942	.88928	.88914	.88901
3100	.95747	.95733	.95719	.95706	.95692	.95678	.95664	.95651	.95637	.95623	8100	.88887	.88873	.88859	.88846	.88832	.88818	.88804	.88791	.88777	.88763
3200	.95610	.95596	.95582	.95568	.95555	.95541	.95527	.95514	.95500	.95486	8200	.88750	.88736	.88722	.88708	.88695	.88681	.88667	.88654	.88640	.88626
3300	.95472	.95459	.95445	.95431	.95418	.95404	.95390	.95376	.95363	.95349	8300	.88612	.88599	.88585	.88571	.88558	.88544	.88530	.88516	.88503	.88489
3400	.95335	.95321	.95308	.95294	.95280	.95267	.95253	.95239	.95225	.95212	8400	.88475	.88461	.88448	.88434	.88420	.88407	.88393	.88379	.88365	.88352
3500	.95198	.95184	.95171	.95157	.95143	.95129	.95116	.95102	.95088	.95075	8500	.88338	.88324	.88311	.88297	.88283	.88269	.88256	.88242	.88228	.88215
3600	.95061	.95047	.95033	.95020	.95006	.94992	.94978	.94965	.94951	.94937	8600	.88201	.88187	.88173	.88160	.88146	.88132	.88118	.88105	.88091	.88077
3700	.94924	.94910	.94896	.94882	.94869	.94855	.94841	.94828	.94814	.94800	8700	.88064	.88050	.88036	.88022	.88009	.87995	.87981	.87968	.87954	.87940
3800	.94786	.94773	.94759	.94745	.94732	.94718	.94704	.94690	.94677	.94663	8800	.87926	.87913	.87899	.87885	.87872	.87858	.87844	.87830	.87817	.87803
3900	.94649	.94635	.94622	.94608	.94594	.94581	.94567	.94553	.94539	.94526	8900	.87789	.87775	.87762	.87748	.87734	.87721	.87707	.87693	.87679	.87666
4000	.94512	.94498	.94485	.94471	.94457	.94443	.94430	.94416	.94402	.94389	9000	.87652	.87638	.87625	.87611	.87597	.87583	.87570	.87556	.87542	.87529
4100	.94375	.94361	.94347	.94334	.94320	.94306	.94292	.94279	.94265	.94251	9100	.87515	.87501	.87487	.87474	.87460	.87446	.87432	.87419	.87405	.87391
4200	.94238	.94224	.94210	.94196	.94183	.94169	.94155	.94142	.94128	.94114	9200	.87378	.87364	.87350	.87336	.87323	.87309	.87295	.87282	.87268	.87254
4300	.94100	.94087	.94073	.94059	.94046	.94032	.94018	.94004	.93991	.93977	9300	.87240	.87227	.87213	.87199	.87186	.87172	.87158	.87144	.87131	.87117
4400	.93963	.93949	.93936	.93922	.93908	.93895	.93881	.93867	.93853	.93840	9400	.87103	.87089	.87076	.87062	.87049	.87035	.87021	.87007	.86993	.86980
4500	.93826	.93812	.93799	.93785	.93771	.93757	.93744	.93730	.93716	.93703	9500	.86966	.86952	.86939	.86925	.86911	.86897	.86884	.86870	.86856	.86843
4600	.93689	.93675	.93661	.93648	.93634	.93620	.93606	.93593	.93579	.93565	9600	.86829	.86815	.86801	.86788	.86774	.86760	.86746	.86733	.86719	.86705
4700	.93552	.93538	.93524	.93510	.93497	.93483	.93469	.93456	.93442	.93428	9700	.86692	.86678	.86664	.86650	.86637	.86623	.86609	.86596	.86582	.86568
4800	.93414	.93401	.93387	.93373	.93360	.93346	.93332	.93318	.93305	.93291	9800	.86554	.86541	.86527	.86513	.86500	.86486	.86472	.86458	.86445	.86431
4900	.93277	.93263	.93250	.93236	.93222	.93209	.93195	.93181	.93167	.93154	9900	.86417	.86403	.86390	.86376	.86362	.86349	.86335	.86321	.86307	.86294
5000	.93140	.93126	.93113	.93099	.93085	.93071	.93058	.93044	.93030	.93017	10000	.86280									

The above table has been taken from *Eggert's Method*, as published in the *U. S. Naval Institute Proceedings*, Vol. 45, Nos. 191 and 200. A table of the function  $H_v$  in metric units is given in *Exterior Ballistic Tables Based on Numerical Integration*, Vol. I, (Ordnance Department, U. S. Army, 1924).

## THE CONSTRUCTION AND USE OF BALLISTIC TABLES BASED ON NUMERICAL INTEGRATION.

701. The principles involved in the computation of trajectories by numerical integration have been covered in the preceding chapter, and the need for ballistic tables in connection with the practical applications of this method also has been touched upon. We shall now examine in greater detail the character of these tables and the features entering into their use.

It is to be noted, first of all, that the physical data which define a trajectory are the velocity-retardation law, the altitude-density law, the value of the gravitational constant  $g$ , and the values of  $\phi$ ,  $V$ , and  $C$ .<sup>\*</sup> Having assumed a definite velocity-retardation law, such as the  $G$ -function, a definite altitude-density law, such as the  $H$ -function, and a definite value of  $g$ , the remaining parameters which define a trajectory are  $\phi$ ,  $V$ , and  $C$ . In modern ballistic tables based on numerical

integration,  $\phi$ ,  $V$ , and  $C$  are therefore the entering arguments. It should not be overlooked, however, that the assumed velocity-retardation law, altitude-density law, and value of  $g$ , are inherent in such tables, and that the tables are strictly applicable only to cases for which these assumptions are correct. Minor variations from these assumptions can be accounted for very conveniently, and with sufficient accuracy for practical purposes, through the medium of the coefficient  $i$  (as outlined in articles 623, 627, and 628). A material variation from these assumptions, as for example a materially different velocity-retardation function, requires a new set of tables, or some process which gives the equivalent thereof.

702. The French were among the first to issue comprehensive ballistic tables based on the modern application of numerical integration. In September, 1918, the *Ministère de l'Armement* issued tables which had been prepared by the *Commission Artillerie Lourde sur Voie Ferrée* (freely translated, Heavy Railway Artillery Commission); in 1921, the tables were extended to include a wider range

of the arguments and some additional elements. These tables were adopted in this country shortly after their first appearance, and have come to be known as the *A.L.V.F. Tables*. The U. S. Navy has continued up until the present time to use these tables extensively in connection with the construction of its range tables for angles of departure above  $15^\circ$ , and we shall, accordingly, devote our present study principally to them.

The *A.L.V.F. Tables* are based on the velocity-retardation relation expressed by the Gâvre function, differing but little from the  $G$ -function (art. 410), and on an altitude-density relation of the same character as the  $H$ -function but differing very slightly from the latter. Numerical integration of the differential equations

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<sup>\*</sup> Certain other factors, such as the forces introduced by the spin of the projectile and the rotation of the earth, as well as wind and motion of gun, might be included here. However, the effects of these influences are always dealt with separately as corrections to the trajectory, as will be seen presently.

derived in Chapter 5 (art. 503), with  $\theta$  taken as the independent variable, was used in computing the tables.

In 1919, the U. S. War Department issued a translation of the first (1918) edition of the A.L.V.F. Tables. This translation involved chiefly the conversion of the argument  $C$  from the French standards to those used in the United States†; metric units for ranges and velocities, as used by the French, were retained in the American translation. Several extracts from the American translation of the 1918 A.L.V.F. Tables are given in Table VI, *Range and Ballistic Tables, 1935*. Sample pages covering the illustrative examples that will be given in this chapter are appended at the end of this chapter.

703. The general arrangement of data in the A.L.V.F. Tables is illustrated in the extracts appended to this chapter, which represent four pages of these tables,  $\phi = 25^\circ$  being the page argument and  $V$  and  $C$  the vertical and horizontal arguments, respectively, for each page. A triple-entry system is obtained by having separate pages for the several tabular values of the argument  $\phi$ , each page having the additional arguments  $V$  and  $C$ . (Compare with the *Azimuth Tables*, in which latitude is the page argument, and hour angle and declination are the vertical and horizontal arguments, respectively, for each page.) In order to accommodate a wide range of values of  $V$  and  $C$  and sufficiently small tabular intervals of each without unduly increasing the size of the page, a number of pages is required for each value of  $\phi$ . In the American edition of the A.L.V.F. Tables each page includes one-half of the total range of values (300 m.s. to 900 m.s.) of the argument  $V$ , and one-fourth of the total range of values (about 3 to about 15) of the argument  $C$ . Also, only one element is tabulated on each page. For each tabular angle of departure there are, therefore, eight pages for each of the tabulated elements ( $X$ ,  $T$ ,  $\omega$ , and  $v_\omega$ ), or thirty-two pages for each tabular value of  $\phi$ . A set of pages as just described is given for each  $5^\circ$  interval in  $\phi$  from  $15^\circ$  to  $40^\circ$  inclusive. The untranslated 1921 edition includes  $\phi = 45^\circ$ ,  $V$  up to 1200 m.s., and  $y_s$  in addition to the elements already noted.

The tabular interval in the argument  $V$  is uniform throughout the tables and is 10 m.s. For the argument  $C$ , both  $C$  itself and its logarithm are given. Since interpolation with respect to  $\text{Log } C$  is both more convenient and more accurate than interpolation with respect to  $C$  itself, a uniform tabular interval of .020 in  $\text{Log } C$  is used; this corresponds to a tabular interval of about 4.7% with respect to  $C$  itself. The odd tabular values of  $C$  are due to the conversion from the French  $C$  of the original tables.

704. The units used in the A.L.V.F. Tables are as follows:

- (a) *Initial velocity* ( $V$ ) is tabulated in meters per second.
- (b) *Striking velocity* ( $v_\omega$ ) is tabulated in decimeters per second; meters per second are obtained by pointing off one place from the right.

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† This conversion accounts principally for the difference between French and American practices as to the manner of defining the coefficient of form and of setting up the ballistic coefficient. Further details may be found in *T.S. No. 148* (U. S. Army, 1921); also p. V, *A.L.V.F. Ballistic Tables* (U. S. Army, 1919), and p. 52, *Report on Computation of Firing Tables for the U. S. Army*, by H. P. Hitchcock.

Units  
used in  
A.L.V.F.  
Tables

- (c) *Range* ( $X$ ) is tabulated in meters.  
 (d) *Time of flight* ( $T$ ) is tabulated in hundredths of a second; seconds are obtained by pointing off two places from the right.  
 (e) *Angle of fall* ( $\omega$ ) is tabulated in half-minutes of arc; degrees and minutes are obtained by dividing by 120 (for degrees) and the remainder by 2 (for minutes).

For example, let us look up the values of the various elements corresponding to the arguments  $\phi = 25^\circ$ ,  $V = 670$  m.s., and  $\text{Log } C = 1.055$  (these may be found in the extracts appended to this chapter). For range we find  $X = 20,875$  meters. For time of flight we find  $T = 49.20$  seconds. For angle of fall we find 4244 half-minutes; dividing 4244 by 120 to get whole degrees, and the remainder by 2 to get minutes, we have  $\omega = 35^\circ 22'$ . For terminal velocity (or striking velocity) we find  $v_\omega = 379.4$  m.s.

705. For converting metric units to English units, and vice versa, we shall use the following relations.\*

one {

$$\begin{aligned} 1 \text{ meter} &= 3.2808 \text{ feet} & (\log 0.51598) \\ 1 \text{ foot} &= .30480 \text{ meter} & (\log 9.48402 - 10) \\ 1 \text{ meter} &= 1.0936 \text{ yard} & (\log 0.03886) \\ 1 \text{ yard} &= .91440 \text{ meter} & (\log 9.96114 - 10) \end{aligned}$$

It is not necessary that the ballistic coefficient  $C$  be found in metric units, since it is a ratio and hence independent of units. The necessary conversion due to the difference between the French ballistic coefficient and our own  $C$  has been made in preparing the translation from the original French tables; the value of  $\text{Log } C$  for use with the translated edition is to be found exactly as explained heretofore.

706. The tabular intervals of the arguments  $V$  and  $C$  in the A.L.V.F. Tables are small enough to make second differences negligible for purposes of interpolation. Ordinary linear interpolation is therefore used with respect to  $V$  and  $C$ . In order to facilitate interpolation with respect to these arguments, differences between adjacent tabular values are recorded alongside the latter in the body of the table. Thus immediately to the right of each value tabulated in the body of the table will be found the difference between the latter and the value next to the right of it, i.e., the difference with respect to adjacent tabular values of the argument  $C$ . Similarly, immediately below the difference with respect to adjacent tabular values of  $C$  will be found the difference with respect to adjacent tabular values of  $V$ . For example, on the page headed  $\phi = 25^\circ$  *Range*, with  $V = 790$  m.s. and  $\text{Log } C = 1.095$ , we find  $X = 27,529$ ; this is the range in meters corresponding to  $\phi = 25^\circ$ ,  $V = 790$  m.s., and  $\text{Log } C = 1.095$ . Immediately to the right of 27,529 we find the number 540, which is the difference between the values of  $X$  for  $\text{Log } C = 1.095$  and for  $\text{Log } C = 1.115$  (28,069 - 27,529), the arguments  $\phi$  and  $V$  having remained the same. Immediately below the number 540 we find the number 516, which is the difference between the values of  $X$  for  $V = 790$  m.s. and for  $V = 800$  m.s. (28,045 - 27,529), the arguments  $\phi$  and  $\text{Log } C$  having remained the same. The arrangement is quite obvious, since the differences with respect to the argument  $C$  appear to the right of the tabular values to which they pertain, this being the direction in which the argument  $C$  increases; while the differences with respect to the argument  $V$  appear below the tabular values to which they pertain, this being the direction in which

\* In the United States the value of the meter is legalized at 39.37 inches; in Great Britain and France, 39.37079 inches are considered to be the equivalent of one meter.

the argument  $V$  increases. Caution must be exercised, however, in the matter of the signs of these differences. The signs are not given in the tabulation but may be determined readily by inspection. It will be noted that  $X$ ,  $T$ , and  $v_\omega$  always increase as either  $V$  or  $C$  increases, while  $\omega$  increases as  $V$  increases but decreases as  $C$  increases.

The following examples cover the various cases of interpolation with respect to  $V$  and  $C$ . The portion of the A.L.V.F. Tables required in connection with these examples is given in the extracts appended to this chapter.

Single interpolation with respect to  $C$       707. I. *Given:*  $\phi = 25^\circ$ ,  $V = 790$  m.s.,  $\text{Log } C = 1.12707$ .  
*Find:* Range, time of flight, angle of fall and terminal velocity (in metric units).

We find  $V = 790$  m.s. is a tabular value, hence this is a case of single interpolation. Then we find, in the table headed  $\phi = 25^\circ$  **Range**,

For  $V = 790$  and  $\text{Log } C = 1.115$ ,  $X = 28,069$

For  $V = 790$  and  $\text{Log } C = 1.135$ ,  $X = 28,607$

The difference between the two values of  $X$  is 538, and is given in the difference column next to 28,069. Then we have, for  $V = 790$  and  $\text{Log } C = 1.12707$ ,

$$X = 28,069 + \frac{.01207}{.020} \times 538 = \underline{28,394 \text{ m}}$$

From the table headed  $\phi = 25^\circ$  **Time of Flight**, we have,

For  $V = 790$  and  $\text{Log } C = 1.115$ ,  $T = 5758$

For  $V = 790$  and  $\text{Log } C = 1.135$ ,  $T = 5796$

The difference between the two values of  $T$  is 38, and is given in the difference column next to 5758. Then we have, for  $V = 790$  and  $\text{Log } C = 1.12707$ ,

$$T = 5758 + \frac{.01207}{.020} \times 38 = 5781 \text{ or } \underline{57.81 \text{ seconds}}$$

Note that the pointing off of decimal places may be done most conveniently after the interpolation has been completed.

From the table headed  $\phi = 25^\circ$  **Angle of Fall**, we have,

For  $V = 790$  and  $\text{Log } C = 1.115$ ,  $\omega = 4379$

For  $V = 790$  and  $\text{Log } C = 1.135$ ,  $\omega = 4335$

The difference between the two values of  $\omega$  is 44, and is given in the difference column next to 4379. Then we have, for  $V = 790$  and  $\text{Log } C = 1.12707$ ,

$$\omega = 4379 - \frac{.01207}{.020} \times 44 = 4352$$

whence we have

$$4352/120 = \underline{36^\circ 16'}$$

Here again it is convenient to complete the interpolation before converting any values into degrees and minutes.

From the table headed  $\phi = 25^\circ$  **Terminal Velocity**, we have,

For  $V = 790$  and  $\text{Log } C = 1.115$ ,  $v_\omega = 4223$

For  $V = 790$  and  $\text{Log } C = 1.135$ ,  $v_\omega = 4296$

The difference between the two values of  $v_\omega$  is 73, and is given in the difference column next to 4223. Then we have, for  $V = 790$  and  $\text{Log } C = 1.12707$ ,

$$v_{\omega} = 4223 + \frac{.01207}{.020} \times 73 = 4267 \text{ or } \underline{426.7 \text{ m.s.}}$$

It will be observed that it is not necessary to write the data for each of the sets of values ( $V=790-\text{Log } C=1.115$  and  $V=790-\text{Log } C=1.135$ ), since the difference column gives us the differences to be used in interpolating. In this first example the data for the two sets of values have been put down in order to make the illustration complete, and in order that there may be no confusion as to the use of the quantities from the difference columns. Normally the entire procedure reduces to the following:

The interpolating ratio is,

$$\frac{1.12707 - 1.115}{.020} = \frac{.01207}{.020} = \underline{.604}$$

whence,

$$\begin{aligned} X &= 28,069 + .604 \times 538 = \dots\dots\dots \underline{28,394 \text{ m.}} \\ T &= 5758 + .604 \times 38 = 5781 \text{ or } \dots\dots\dots \underline{57.81 \text{ sec.}} \\ \omega &= 4379 - .604 \times 44 = 4352 \text{ or } \dots\dots\dots \underline{36^{\circ} 16'} \\ v_{\omega} &= 4223 + .604 \times 73 = 4267 \text{ or } \dots\dots\dots \underline{426.7 \text{ m.s.}} \end{aligned}$$

Single  
interpolation  
with  
respect to  $V$

708. II. *Given:*  $\phi = 25^{\circ}$ ,  $V = 792.5 \text{ m.s.}$ ,  $\text{Log } C = 1.11500$ .

*Find:* Range, time of flight, angle of fall and terminal velocity (metric units).

In this case we find that  $\text{Log } C = 1.11500$  is a tabular value, hence this also is a case of single interpolation. Remembering that the differences with respect to  $V$  appear below those with respect to  $C$ , we will proceed directly to the abbreviated process just illustrated in the last example.

The interpolating ratio is,

$$\frac{792.5 - 790}{10} = .25 \text{ or } \frac{1}{4}$$

whence,

$$\begin{aligned} X &= 28,069 + \frac{1}{4} \times 532 = \dots\dots\dots \underline{28,202 \text{ m.}} \\ T &= 5758 + \frac{1}{4} \times 61 = 5773 \text{ or } \dots\dots\dots \underline{57.73 \text{ sec.}} \\ \omega &= 4379 + \frac{1}{4} \times 17 = 4383 \text{ or } \dots\dots\dots \underline{36^{\circ} 32'} \\ v_{\omega} &= 4223 + \frac{1}{4} \times 21 = 4228 \text{ or } \dots\dots\dots \underline{422.8 \text{ m.s.}} \end{aligned}$$

Double  
interpolation

709. III. *Given:*  $\phi = 25^{\circ}$ ,  $V = 792.5 \text{ m.s.}$ ,  $\text{Log } C = 1.12707$ .

*Find:* Range, time of flight, angle of fall, and terminal velocity (metric units).

Neither  $V$  nor  $C$  are tabular values, hence this is a case of double interpolation. We shall proceed by making the interpolation for  $C$  for the velocities next lower and next higher to the given velocity, and then interpolating between these values for the given velocity.

For  $\text{Log } C$  the interpolating ratio is,

$$\frac{1.12707 - 1.115}{.020} = \underline{.604}$$

For  $V$  the interpolating ratio is,

$$\frac{792.5 - 790}{10} = \frac{1}{4}$$

Then we have, interpolating first for  $\text{Log } C$  and then for  $V$ ,

$$\text{For } V = 790, \quad X = 28,069 + .604 \times 538 = 28,394$$

$$\text{For } V = 800, \quad X = 28,601 + .604 \times 554 = 28,936$$

$$\begin{aligned} \text{For } V = 792.5, \quad X &= 28,394 + \frac{1}{4}(28,936 - 28,394) \\ &= 28,394 + \frac{1}{4} \times 542 = \underline{28,530 \text{ m.}} \end{aligned}$$

$$\text{For } V = 790, \quad T = 5758 + .604 \times 38 = 5781$$

$$\text{For } V = 800, \quad T = 5819 + .604 \times 38 = 5842$$

$$\begin{aligned} \text{For } V = 792.5, \quad T &= 5781 + \frac{1}{4}(5842 - 5781) \\ &= 5781 + \frac{1}{4} \times 61 = 5796 \text{ or } \underline{57.96 \text{ sec.}} \end{aligned}$$

$$\text{For } V = 790, \quad \omega = 4379 - .604 \times 44 = 4352$$

$$\text{For } V = 800, \quad \omega = 4396 - .604 \times 45 = 4369$$

$$\begin{aligned} \text{For } V = 792.5, \quad \omega &= 4352 + \frac{1}{4}(4369 - 4352) \\ &= 4352 + \frac{1}{4} \times 17 = 4356 \text{ or } \underline{36^\circ 18'} \end{aligned}$$

$$\text{For } V = 790, \quad v_\omega = 4223 + .604 \times 73 = 4267$$

$$\text{For } V = 800, \quad v_\omega = 4244 + .604 \times 75 = 4289$$

$$\begin{aligned} \text{For } V = 792.5, \quad v_\omega &= 4267 + \frac{1}{4}(4289 - 4267) \\ &= 4267 + \frac{1}{4} \times 22 = 4273 \text{ or } \underline{427.3 \text{ m.s.}} \end{aligned}$$

710. The above examples cover cases of entry into the A.L.V.F. Tables with the usual arguments, viz.,  $\phi$ ,  $V$ , and  $C$ . Occasion arises also for determining the value of the argument  $C$  that corresponds to given values of  $\phi$ ,  $V$ , and  $X$  (as noted in article 624). This constitutes indirect entry into the tables, which is illustrated by the following example.

Entry into A.L.V.F. Tables, with  $\phi$ ,  $V$ , and  $X$  as arguments to find  $\text{Log } C$ .

IV. *Given:*  $\phi = 25^\circ$ ,  $V = 791 \text{ m.s.}$ ,  $X = 28,908 \text{ m.}$

*Find:*  $\text{Log } C$ .

Examining the table headed  $\phi = 25^\circ$  **Range**, we find that, in the vicinity of  $V = 790$ , the value  $X = 28,908$  lies between the columns headed  $\text{Log } C = 1.135$  and  $\text{Log } C = 1.155$ . We will first find  $X$ , for the given value of  $V = 791$ , in each of these columns.

$$\text{For } \text{Log } C = 1.135, \quad X = 28,607 + .1 \times 548 = 28,662$$

$$\text{For } \text{Log } C = 1.155, \quad X = 29,143 + .1 \times 565 = 29,199$$

Both of the values of  $X$  we have found are for  $V = 791$ ; since our given value of  $X = 28,908$  lies between them, we may find the value of  $\text{Log } C$  corresponding to it by the usual interpolation process. We have,

$$\begin{aligned} \text{Log } C &= 1.135 + \frac{28,908 - 28,662}{29,199 - 28,662} \times (1.155 - 1.135) \\ &= 1.135 + \frac{246}{537} \times .020 = \underline{1.14416} \end{aligned}$$

711. The tabular interval in  $\phi$  in the A.L.V.F. tables is too great for linear interpolation with respect to this argument, but the required accuracy may be obtained by extending the interpolation to second differences. When an entire range table is being prepared, and the values of the various elements have been found for the several tabular values of  $\phi$  (i.e.,  $15^\circ$ ,  $20^\circ$ ,  $25^\circ$ ,  $30^\circ$ ,  $35^\circ$ , and  $40^\circ$ ), the problem of interpolation for values of these elements between the tabular values of  $\phi$  can be handled about

Interpolation with respect to  $\phi$  in A.L.V.F. Tables

as satisfactorily by graphical methods as by interpolation by formulas involving the higher differences. In constructing U. S. Navy range tables from the A.L.V.F. Tables, the graphical process is used. The values of the several elements are found from the A.L.V.F. Tables for each  $5^\circ$  interval in  $\phi$  from  $15^\circ$  to  $40^\circ$ , and are plotted against  $\phi$ . Fair curves are passed through the plotted points and the values at any number of required intermediate points may then be picked from the curves. Regularity in the intermediate values so obtained is secured both by using an appropriately large scale for the plotting, and by adjusting the values picked from the curves so that the second differences run smoothly. This graphical process is practically the equivalent of interpolation with second differences, and is more convenient when a great many intermediate values are to be found.

712. For isolated cases involving the determination of elements for other than tabular values of  $\phi$ , the usual methods of interpolation with second differences may be applied. The formula for interpolation with second differences is\*

Use of second differences in interpolating with respect to  $\phi$

$$= f_0 + m\Delta_1 + \frac{m(m-1)}{2} \Delta_2 \quad (701)$$

held for this  
 in which  $f$  is the value of the element to be found;  $f_0$  is the next lower tabular value of this element;  $\Delta_1$  and  $\Delta_2$  are, respectively, the first and second differences for this element with respect to the argument against which the interpolation is to be made; and  $m$  is the interpolating ratio with respect to the same argument, i.e., the proportion of its tabular interval for which the interpolation is to be made. Due consideration must be given to the signs of both the first and second differences; also, since  $m$  is always less than unity, it is to be noted that the quantity  $(m-1)$  is always negative.

For illustration we shall use the following example, taking tabular values for  $V$  and  $C$  in order to confine the interpolation to the argument requiring the use of second differences.

Given:  $\phi = 23\frac{1}{2}^\circ$ ,  $V = 790$  m.s.,  $\text{Log } C = 1.13500$ .

Find: Range (metric units).

In order to find the required first and second differences with respect to  $\phi$  we must find  $X$  for three successive values of  $\phi$ , and we shall therefore find  $X$  for  $\phi = 20^\circ$ ,  $25^\circ$ , and  $30^\circ$ , using  $V = 790$  m.s. and  $\text{Log } C = 1.13500$  in each case. Tabulating these values of  $X$ , the differences are found readily as follows.

$\phi$	$X$	$V = 790$ m.s.		$\text{Log } C = 1.135$	
		$\Delta_1$	$\Delta_2$	$\Delta_1$	$\Delta_2$
$20^\circ$	25,130				
		+3477			
$25^\circ$	28,607		-603		
		+2874			
$30^\circ$	31,481				

The interpolating ratio is  $\frac{3.5^\circ}{5^\circ}$ , or  $m = 0.7$ ; the value of  $X$  for the tabular value

$\phi = 20^\circ$  (which is the next lower to  $23\frac{1}{2}^\circ$ ) gives  $f_0 = 25,130$ ; the first difference corresponding to  $f_0$  is  $\Delta_1 = +3477$ , and the second difference is  $\Delta_2 = (-)603$ . Sub-

\* Ref. p. 115, Marks' *Mechanical Engineers' Handbook*, Second Edition.

stituting these values in (701) we have,

$$f(\text{or } X) = 25,130 + 0.7 \times 3477 + 0.105 \times 603 = 27,627 \text{ m.}$$

Had  $V$  and  $C$  not been tabular values, it would have been necessary to determine  $X$  for each of the three tabular values of  $\phi$  by interpolation just as already illustrated in article 709. The remaining process would then have been reduced to that shown above. Interpolation with respect to  $\phi$  for the elements  $T$ ,  $\omega$ , and  $v_\omega$  involves nothing different than shown in the above example for  $X$ .

713. If it should be necessary to find the value of  $C$  corresponding to given values of  $X$ ,  $V$ , and  $\phi$ , with  $\phi$  not a tabular value, we proceed as illustrated in the following example.

*Given:*  $\phi = 23\frac{1}{2}^\circ$ ,  $V = 790$  m.s.,  $X = 27,754$  m.

*Find:* The value of  $\text{Log } C$  corresponding to the above values.

Indirect entry  
into A.L.V.F.  
Tables when  
 $\phi$  is not a  
tabular value

By trial and error we find that the required  $\text{Log } C$  lies between the tabular values 1.135 and 1.155; hence we shall find  $X$  for each of these values of  $\text{Log } C$ , and with  $\phi = 23\frac{1}{2}^\circ$  and  $V = 790$  m.s. in each case. Then for

$\phi$	$V = 790 \text{ m.s.}$ $X$	$\Delta_1$	$\Delta_2$
$20^\circ$	25,130		
		+3477	
$25^\circ$	28,607		-603
		+2874	
$30^\circ$	31,481		

whence for  $\phi = 23\frac{1}{2}^\circ$ ,  $X = 27,627$  m. (as already found in the preceding example). Also for

$\phi$	$V = 790 \text{ m.s.}$ $X$	$\Delta_1$	$\Delta_2$
$20^\circ$	25,570		
		+3573	
$25^\circ$	29,143		-611
		+2962	
$30^\circ$	32,105		

whence for  $\phi = 23\frac{1}{2}^\circ$ ,  $X = 25,570 + 0.7 \times 3573 + 0.105 \times 611 = 28,135$  m.

We now have two values of  $X$  as follows,

$$\phi = 23\frac{1}{2}^\circ, V = 790 \text{ m.s., } \text{Log } C = 1.135, X = 27,627 \text{ m.}$$

$$\phi = 23\frac{1}{2}^\circ, V = 790 \text{ m.s., } \text{Log } C = 1.155, X = 28,135 \text{ m.}$$

The given value  $X = 27,754$  m. lies between these, and we interpolate for the corresponding  $\text{Log } C$  just as in article 710, thus

$$\begin{aligned} \text{Log } C &= 1.135 + \frac{27,754 - 27,627}{28,135 - 27,627} \times .020 \\ &= 1.135 + \frac{127}{508} \times .020 = \underline{1.140} \end{aligned}$$

Had  $V$  not been a tabular value, the process merely would have been lengthened by the requirement of interpolating with respect to  $V$  for each of the adjacent values of  $\text{Log } C$ .

714. A few years after the French issued their A.L.V.F. Tables, the U. S. War Department commenced the preparation of a much more comprehensive set of tables of its own, under the title *Exterior Ballistic Tables Based on Numerical Integration*, to which we shall refer hereafter as the *War Department Tables*. These tables are based on numerical integration of the differential equations that have been derived in article 610, with  $t$  as the independent variable. The  $G$ -function and  $H$ -function, and  $g = 9.80$  m.s.s., are employed, and the process of solution is similar to that illustrated in articles 611-620 and further outlined in articles 621-622.

The War Department Tables, when completed, will include three volumes. Volume I, issued in 1924, is designed to afford a means of determining numerous points of any trajectory likely to be encountered in practice, and is based on the principles outlined in article 630. Volume II, issued in 1931, is required in connection with Volume I; it gives the values of the summital elements  $x_s, y_s, t_s, v_s$ , in terms of the arguments  $\phi, V$ , and  $C$ . Volume III, at the time of this writing, is still in the process of preparation; it will give the values of the terminal elements  $X, T, \omega, v_\omega$ , in terms of the arguments  $\phi, V$ , and  $C$ . Volume III of the War Department Tables is quite similar in purpose to the A.L.V.F. Tables, although the arrangement is somewhat different.

The scope of the War Department Tables is  $0^\circ$  to  $90^\circ$  in  $\phi$ , 80 m.s. to 1000 m.s. in  $V$ , and 0.000 to 1.250 in  $\text{Log } C$ . The tabular intervals for  $V$  and  $C$  vary among the several volumes but are in all cases greater than in the A.L.V.F. Tables. The tabular interval for  $\phi$  is  $1^\circ$  from  $\phi = 0^\circ$  to  $\phi = 30^\circ$ , and  $2^\circ$  from  $\phi = 30^\circ$  to  $\phi = 90^\circ$ , in all cases. The greater scope in values of  $\phi$  and the smaller tabular intervals therein, as compared to the A.L.V.F. Tables, constitute important advantages over the latter. The War Department Tables are applicable to all angles of departure; furthermore, the tabular intervals for all of the arguments ( $\phi, V$ , and  $C$ ) are small enough to render linear interpolation with respect to any of them satisfactory for ordinary problems.\* (While second differences with respect to  $V$  are not altogether negligible, the error that results from neglecting them rarely exceeds a very few units in the last significant figure retained and is of little importance, especially in connection with the finding of the maximum ordinate, to which our immediate use of these tables is to be confined.)

715. Since the American edition of the A.L.V.F. Tables does not include the maximum ordinate, we shall use for this element the appropriate portions of Volume II of the War Department Tables. Extracts of sufficient scope to cover all exercises dealt with in this text are given in Table VII, *Range and Ballistic Tables, 1935*. Sample pages covering the illustrative examples that will be given in this chapter are appended at the end of this chapter.

Referring to Table VII, it is to be noted that the most important difference

\* It is the author's understanding, at the time of this writing, that when Volume III of the War Department Tables is issued it will be adopted by the U. S. Navy Department for the preparation of its range tables, replacing the Ingalls-Siacci Method now in use for angles of departure up to  $15^\circ$  and the A.L.V.F. Tables now in use for angles of departure above  $15^\circ$ . Although Volumes I and II by themselves afford a complete solution for any required trajectory, the process involved in connection with the use of these volumes is rather laborious as compared to the methods now in use, except in the case of finding the maximum ordinate from Volume II. Any further explanation of the arrangement or use of Volume I would lead to a digression that is not warranted, since no application of this volume is to be used in this text (nor is it, indeed, used by the Navy Department). The limited use of Volume II that is to be made in this text affords ample illustration of the character of this volume, and also a fairly good illustration of the character of Volume III.

between these tables and Table VI (A.L.V.F. Tables) is that in the former the page argument is  $V$ , and the vertical argument is  $\phi$ . With the additional exception that the difference columns are not given in Table VII, the arrangement of the two tables is very similar. Further points to be noted in connection with Table VII are that it is in metric units, that the tabular interval in  $V$  is 40 m.s., and that the tabular interval in  $\text{Log } C$  is .050.

716. Interpolation in Table VII offers no new problems. It is to be noted that interpolation with respect to the page argument  $V$  will ordinarily be required. However, this does not involve the complications met in interpage interpolation in the A.L.V.F. Tables, since in Table VII linear interpolation may be used between pages (i.e., with respect to  $V$ ), as well as on each page (i.e., with respect to  $\phi$  and  $C$ ).

For illustration of the use of Table VII we shall take the same data as given in the example in article 709.

*Given:*  $\phi = 25^\circ$ ,  $V = 792.5$  m.s.,  $\text{Log } C = 1.12707$ .

*Find:* Maximum ordinate (metric units).

Use of the War  
Department  
Tables for  
finding the  
maximum  
ordinate

The given  $V$  lies between the tabular values 760 m.s. and 800 m.s.; we shall proceed by finding  $y_s$  for each of these values of  $V$  and with the given  $\phi$  and  $\text{Log } C$ . Since  $\phi$  is a tabular value this involves interpolation only with respect to  $\text{Log } C$ , for which the

interpolating ratio is

$$\frac{1.12707 - 1.10000}{.050} = \underline{.541}.$$

Performing the required interpolation on the pages headed  $V = 760$  m.s. and  $V = 800$  m.s. we have, for  $\phi = 25^\circ$  and  $\text{Log } C = 1.12707$  in each case,

$$\text{For } V = 760 \text{ m.s., } y_s = 3815 + .541 \times 111 = 3875$$

$$\text{For } V = 800 \text{ m.s., } y_s = 4153 + .541 \times 127 = 4222$$

The interpolating ratio for  $V$  is

$$\frac{792.5 - 760}{40} = \underline{.812}$$

whence we have

$$\text{For } V = 792.5 \text{ m.s., } y_s = 3875 + .812 \times 347 = \underline{4157\text{m.}}$$

## EXERCISES

1. *Given:* The values of  $\phi$ ,  $V$ , and  $\text{Log } C$  (metric units).

*Find:* The values of  $X$ ,  $T$ ,  $\omega$ ,  $v_\omega$ , and  $y_s$  (metric units).

	Given			Answers				
	$\phi$	$V$ (m.s.)	$\text{Log } C$	$X$ (m.)	$T$ (sec.)	$\omega$	$v_\omega$ (m.s.)	$y_s$ (m.)
A	$15^\circ$	800.0	1.11500	21,067	37.30	$21^\circ 48'$	449.1	1715
B	25	680.0	1.12500	22,633	50.89	34 33	400.8	3210
C	30	612.5	1.01500	19,560	52.60	40 22	363.9	3463
D	40	791.6	1.11590	34,828	85.30	52 00	453.9	9000
E	25	883.9	1.01600	29,686	60.74	39 50	403.3	4599

2. *Given:* The values of  $\phi$ ,  $V$ , and  $\text{Log } C$  (English units).

*Find:* The values of  $X$ ,  $T$ ,  $\omega$ ,  $v_\omega$  and  $y_s$  (English units). (Use  $V$  to the nearest tenth of a m.s.)

	Given			Answers				
	$\phi$	$V$ (f.s.)	$\text{Log } C$	$X$ (yds.)	$T$ (sec.)	$\omega$	$v_\omega$ (f.s.)	$y_s$ (ft.)
A	15°	2625	1.11500	23,043	37.30	21° 48'	1473	5626
B	25	2231	1.12500	24,751	50.89	34 33	1315	10,532
C	30	2010	1.01500	21,401	52.62	40 22	1194	11,368
D	40	2597	1.11590	38,088	85.30	52 00	1489	29,527
E	25	2900	1.01600	32,465	60.74	39 50	1323	15,092

3. *Given:* The values of  $\phi$ ,  $V$ , and  $X$  (English units).

*Find:* The values of  $\text{Log } C$  and  $\omega$ . (Use  $V$  to the nearest tenth of a m.s.)

	Given			Answers	
	$\phi$	$V$ (f.s.)	$X$ (yds.)	$\text{Log } C$	$\omega$
A	15°	2625	23,043	1.11500	21° 48'
B	25	2231	24,752	1.12500	34 33
C	30	2010	21,401	1.01500	40 22
D	40	2597	38,088	1.11590	52 00
E	25	2900	32,465	1.01600	39 50

$\phi = 25^\circ$ . RANGE.

Log. C (c)	1.015 (10.351) dif.	1.035 (10.839) dif.	1.055 (11.350) dif.	1.075 (11.885) dif.	1.095 (12.445) dif.	1.115 (13.032) dif.	1.135 (13.646) dif.	1.155 (14.289) dif.	1.175 (14.962) dif.	Log. C (c)
V. 600	17305 270 396	17575 270 408	17845 270 421	18115 269 434	18384 267 446	18651 266 459	18917 265 472	19182 264 485	19446 498	600
610	17701 282 399	17983 283 412	18266 283 425	18549 281 438	18830 280 451	19110 279 464	19389 278 477	19667 277 490	19944 503	610
620	18100 295 403	18395 296 416	18691 296 429	18987 294 442	19281 293 455	19574 292 468	19866 291 481	20157 290 494	20447 507	620
630	18503 308 408	18811 309 421	19120 309 433	19429 307 446	19736 306 460	20042 305 473	20347 304 486	20651 303 499	20954 512	630
640	18911 321 411	19232 321 424	19553 322 437	19875 321 450	20196 319 463	20515 318 476	20833 317 489	21150 316 503	21466 517	640
650	19322 334 415	19656 334 428	19990 335 441	20325 334 454	20659 332 467	20991 331 480	21322 331 494	21653 330 508	21983 522	650
660	19737 347 417	20084 347 430	20431 348 444	20779 347 458	21126 345 472	21471 345 486	21816 345 500	22161 344 513	22505 526	660
670	20154 360 420	20514 361 434	20875 362 448	21237 361 462	21598 359 475	21957 359 490	22316 358 503	22674 357 517	23031 531	670
680	20574 374 423	20948 375 437	21323 376 452	21699 374 466	22073 374 480	22447 372 493	22819 372 507	23191 371 521	23562 535	680
690	20997 388 426	21385 390 441	21775 390 455	22165 388 469	22553 387 484	22940 386 498	23326 386 512	23712 385 526	24097 540	690
700	21423 403 427	21826 404 441	22230 404 456	22634 403 471	23037 401 486	23438 400 501	23838 400 516	24238 399 530	24637 544	700
710	21850 417 431	22267 419 446	22686 419 460	23105 418 474	23523 416 488	23939 415 503	24354 414 518	24768 413 533	25181 548	710
720	22281 432 434	22713 433 448	23146 433 463	23579 432 478	24011 431 493	24442 430 508	24872 429 523	25301 428 538	25729 552	720
730	22715 446 435	23161 448 451	23609 448 466	24057 447 481	24504 446 496	24950 445 511	25395 444 526	25839 442 541	26281 556	730
740	23150 462 436	23612 463 452	24075 463 468	24538 462 484	25000 461 499	25461 460 514	25921 459 529	26380 457 544	26837 560	740
750	23586 478 438	24064 479 455	24543 479 471	25022 477 487	25499 476 503	25975 475 519	26450 474 534	26924 473 549	27397 564	750
760	24024 495 441	24519 495 458	25014 495 474	25509 493 490	26002 492 506	26494 490 522	26984 489 538	27473 488 553	27961 568	760
770	24465 512 444	24977 511 461	25488 511 477	25999 509 493	26508 508 509	27016 506 525	27522 504 541	28026 503 557	28529 573	770
780	24909 529 446	25438 527 463	25965 527 480	26492 525 496	27017 524 512	27541 522 528	28063 520 544	28583 519 560	29102 577	780
790	25355 546 448	25901 544 466	26445 543 483	26988 541 499	27529 540 516	28069 538 532	28607 536 548	29143 536 565	29679 581	790
800	25803 564 451	26367 561 467	26928 559 485	27487 558 502	28045 556 519	28601 554 536	29155 553 552	29708 552 569	30260 585	800
810	26254 580 453	26834 579 469	27413 567 487	27989 575 504	28564 573 521	29137 570 538	29707 570 555	30277 568 572	30845 589	810
820	26707 596 454	27303 597 471	27900 593 488	28493 592 506	29085 590 523	29675 587 540	30262 587 558	30849 585 576	31434 594	820
830	27161 613 458	27774 614 475	28388 611 492	28999 609 509	29608 607 526	30215 605 544	30820 605 562	31425 603 580	32028 598	830
840	27619 630 459	28249 631 475	28880 628 493	29508 626 512	30134 625 530	30759 623 548	31382 623 566	32005 621 584	32626 603	840
850	28078 646 462	28724 649 479	29373 647 496	30020 644 514	30664 643 533	31307 641 551	31948 641 569	32589 640 588	33229 607	850
860	28540 663 463	29203 666 481	29869 665 499	30534 663 517	31197 661 536	31858 659 554	32517 660 573	33177 659 592	33836 611	860
870	29003 681 465	29684 684 483	30368 683 501	31051 682 520	31733 679 539	32412 678 558	33090 679 578	33769 678 597	34447 616	870
880	29468 699 469	30167 702 487	30869 702 505	31571 701 523	32272 698 542	32970 698 562	33668 698 582	34366 697 601	35063 620	880
890	29937 717 471	30654 720 489	31374 720 507	32094 720 526	32814 718 545	33532 718 565	34250 717 585	34967 716 605	35683 625	890
900	30408 735	31143 738	31881 739	32620 739	33359 738	34097 738	34835 737	35572 736	36308	900
V. Log. C	(c) 1.015	(c) 1.035	(c) 1.055	(c) 1.075	(c) 1.095	(c) 1.115	(c) 1.135	(c) 1.155	(c) 1.175	(c) Log. C

$\phi = 25^\circ$ . TIME OF FLIGHT.

Log. C (c)	1.015 (10.351) dif.	1.035 (10.839) dif.	1.055 (11.350) dif.	1.075 (11.885) dif.	1.095 (12.445) dif.	1.115 (13.032) dif.	1.135 (13.646) dif.	1.155 (14.289) dif.	1.175 (14.962) dif.	Log. C (c)
V. 600	4440 26 60	4466 26 61	4492 25 62	4517 24 62	4541 23 63	4564 23 64	4587 22 65	4609 22 66	4631 67	600
610	4500 27 60	4527 27 61	4554 25 61	4579 25 62	4604 24 63	4628 24 64	4652 23 65	4675 23 66	4698 67	610
620	4560 28 59	4588 27 60	4615 26 61	4641 26 62	4667 25 63	4692 25 64	4717 24 65	4741 24 66	4765 66	620
630	4619 29 59	4648 28 60	4676 27 61	4703 27 62	4730 26 63	4756 26 64	4782 25 65	4807 24 65	4831 65	630
640	4678 30 59	4708 29 60	4737 28 61	4765 28 62	4793 27 63	4820 27 64	4847 25 64	4872 24 65	4896 66	640
650	4737 31 59	4768 30 60	4798 29 61	4827 29 62	4856 28 63	4884 27 64	4911 26 65	4937 25 65	4962 65	650
660	4796 32 59	4828 31 60	4859 30 61	4889 30 62	4919 29 63	4948 28 64	4976 26 64	5002 25 65	5027 66	660
670	4855 33 59	4888 32 60	4920 31 61	4951 31 62	4982 30 63	5012 28 63	5040 27 64	5067 26 65	5093 66	670
680	4914 34 59	4948 33 60	4981 32 61	5013 32 62	5045 30 62	5075 29 63	5104 28 64	5132 27 65	5159 66	680
690	4973 35 59	5008 34 59	5042 33 60	5075 32 61	5107 31 62	5138 30 63	5168 29 64	5197 28 65	5225 65	690
700	5032 35 59	5067 35 60	5102 34 60	5136 33 61	5169 32 62	5201 31 62	5232 30 63	5262 28 64	5290 65	700
710	5091 36 59	5127 35 60	5162 35 61	5197 34 61	5231 32 62	5263 32 63	5295 31 63	5326 29 64	5355 65	710
720	5150 37 59	5187 36 59	5223 35 60	5258 35 61	5293 33 61	5326 32 62	5358 32 63	5390 30 64	5420 65	720
730	5209 37 58	5246 37 59	5283 36 60	5319 35 61	5354 34 62	5388 33 62	5421 33 63	5454 31 63	5485 64	730
740	5267 38 58	5305 38 59	5343 37 60	5380 36 61	5416 34 61	5450 34 62	5484 33 63	5517 32 64	5549 64	740
750	5325 39 57	5364 39 58	5403 38 59	5441 36 60	5477 35 61	5512 35 62	5547 34 62	5581 32 63	5613 64	750
760	5382 40 57	5422 40 58	5462 39 59	5501 37 60	5538 36 61	5574 35 62	5609 35 63	5644 33 63	5677 64	760
770	5439 41 57	5480 41 58	5521 40 59	5561 38 60	5599 37 61	5636 36 61	5672 35 62	5707 34 62	5741 63	770
780	5496 42 57	5538 42 58	5580 41 59	5631 39 59	5660 37 60	5697 37 61	5734 35 62	5769 35 63	5804 63	780
790	5553 43 57	5596 43 58	5639 41 58	5680 40 59	5720 38 60	5758 38 61	5796 36 61	5832 35 62	5867 63	790
800	5610 44 56	5654 43 57	5697 42 57	5739 41 58	5780 39 59	5819 38 60	5857 37 60	5894 36 61	5930 62	800
810	5666 45 55	5711 43 56	5754 43 57	5797 42 57	5839 40 58	5879 38 59	5917 38 60	5955 37 60	5992 61	810
820	5721 46 54	5767 44 55	5811 43 56	5854 43 57	5897 41 57	5938 39 58	5977 38 59	6015 38 60	6053 61	820
830	5775 47 54	5822 45 55	5867 44 56	5911 43 56	5954 42 57	5996 40 57	6036 39 58	6075 39 59	6114 60	830
840	5829 48 54	5877 46 55	5923 44 56	5967 44 57	6011 42 57	6053 41 58	6094 40 59	6134 40 60	6174 61	840
850	5883 49 55	5932 47 56	5979 45 57	6024 44 58	6068 43 58	6111 42 59	6153 41 60	6194 41 61	6235 62	850
860	5938 50 55	5988 48 56	6036 46 57	6082 44 58	6126 44 59	6170 43 60	6213 42 61	6255 42 62	6297 63	860
870	5993 51 56	6044 49 57	6093 47 58	6140 45 59	6185 45 60	6230 44 61	6274 43 62	6317 43 63	6360 63	870
880	6049 52 57	6101 50 58	6151 48 59	6199 46 60	6245 46 61	6291 45 61	6336 44 62	6380 43 63	6423 64	880
890	6106 53 58	6159 51 58	6210 49 59	6259 47 60	6306 46 61	6352 46 62	6398 45 63	6443 44 64	6487 65	890
900	6164 53	6217 52	6269 50	6319 48	6367 47	6414 47	6461 46	6507 45	6552	900
V. Log. C	(10.351) dif. 1.015	(10.839) dif. 1.035	(11.350) dif. 1.055	(11.885) dif. 1.075	(12.445) dif. 1.095	(13.032) dif. 1.115	(13.646) dif. 1.135	(14.289) dif. 1.155	(14.962) dif. 1.175	(c) Log. C

$\phi = 25^\circ$ . ANGLE OF FALL.

Log. C (c) V.	1.015 (10.351) dif.	1.035 (10.839) dif.	1.055 (11.350) dif.	1.075 (11.885) dif.	1.095 (12.445) dif.	1.115 (13.032) dif.	1.135 (13.646) dif.	1.155 (14.289) dif.	1.175 (14.962) dif.	Log. C (c) V.
600	4109 25 31	4084 26 30	4058 27 29	4031 27 27	4004 27 25	3977 28 24	3949 29 23	3920 30 23	3890 22	600
610	4140 26 30	4114 27 29	4087 29 28	4058 29 27	4029 28 26	4001 29 24	3972 29 23	3943 31 22	3912 22	610
620	4170 27 29	4143 28 28	4115 30 27	4085 30 26	4055 30 25	4025 30 24	3995 30 23	3965 31 22	3934 22	620
630	4199 28 29	4171 29 28	4142 31 26	4111 31 26	4080 31 25	4049 31 24	4018 31 23	3987 31 22	3956 21	630
640	4228 29 28	4199 31 27	4168 31 26	4137 32 25	4105 32 24	4073 32 24	4041 32 23	4009 32 22	3977 21	640
650	4256 30 27	4226 32 26	4194 32 25	4162 33 24	4129 32 24	4097 33 23	4064 33 22	4031 33 21	3998 20	650
660	4283 31 27	4252 33 26	4219 33 25	4186 33 24	4153 33 23	4120 34 22	4086 34 22	4052 34 21	4018 20	660
670	4310 32 26	4278 34 25	4244 34 24	4210 34 24	4176 34 23	4142 34 22	4108 35 21	4073 35 21	4038 20	670
680	4336 33 26	4303 35 25	4268 34 24	4234 35 23	4199 35 23	4164 35 22	4129 35 21	4094 36 20	4058 19	680
690	4362 34 25	4328 36 24	4292 35 24	4257 35 23	4222 36 22	4186 36 22	4150 36 21	4114 37 20	4077 19	690
700	4387 35 26	4352 36 25	4316 36 24	4280 36 23	4244 36 22	4208 37 21	4171 37 20	4134 38 20	4096 19	700
710	4413 36 25	4377 37 24	4340 37 23	4303 37 22	4266 37 21	4229 38 20	4191 37 19	4154 39 19	4115 18	710
720	4438 37 25	4401 38 24	4363 38 23	4325 38 22	4287 38 21	4249 39 20	4210 37 19	4173 40 18	4133 17	720
730	4463 38 24	4425 39 23	4386 39 22	4347 39 21	4308 39 20	4269 40 19	4229 38 18	4191 41 17	4150 16	730
740	4487 39 23	4448 40 22	4408 40 21	4368 40 20	4328 40 20	4288 41 19	4247 39 18	4208 42 17	4166 16	740
750	4510 40 22	4470 41 22	4429 41 21	4388 40 20	4348 41 19	4307 42 19	4265 40 18	4225 43 17	4182 16	750
760	4532 40 22	4492 42 21	4450 42 21	4408 41 20	4367 41 19	4326 43 18	4283 41 18	4242 44 17	4198 16	760
770	4554 41 22	4513 42 21	4471 43 20	4428 42 20	4386 42 19	4344 43 18	4301 42 17	4259 45 16	4214 16	770
780	4576 42 21	4534 43 20	4491 43 20	4448 43 19	4405 43 18	4362 44 17	4318 43 17	4275 45 16	4230 15	780
790	4597 43 21	4554 43 20	4511 44 19	4467 44 19	4423 44 18	4379 44 17	4335 44 16	4291 46 15	4245 15	790
800	4618 44 21	4574 44 21	4530 44 20	4486 45 19	4441 45 19	4396 45 18	4351 45 17	4306 46 16	4260 15	800
810	4639 44 21	4595 45 21	4550 45 20	4505 45 19	4460 46 18	4414 46 17	4368 46 16	4322 47 15	4275 15	810
820	4660 44 21	4616 46 20	4570 46 19	4524 46 18	4478 47 17	4431 47 17	4384 47 16	4337 47 15	4290 14	820
830	4681 45 21	4636 47 19	4589 47 19	4542 47 18	4495 47 17	4448 48 17	4400 48 16	4352 48 15	4304 14	830
840	4702 47 19	4655 47 19	4608 48 18	4560 48 17	4512 47 17	4465 49 16	4416 49 15	4367 49 14	4318 13	840
850	4721 47 19	4674 48 18	4626 49 17	4577 48 16	4529 48 15	4481 50 14	4431 50 14	4381 50 13	4331 12	850
860	4740 48 18	4692 49 18	4643 50 17	4593 49 16	4544 49 15	4495 50 14	4445 51 14	4394 51 13	4343 12	860
870	4758 48 18	4710 50 17	4660 51 16	4609 50 16	4559 50 15	4509 50 14	4459 52 13	4407 52 12	4355 12	870
880	4776 49 17	4727 51 16	4676 51 15	4625 51 15	4574 51 14	4523 51 13	4472 53 12	4419 52 12	4367 11	880
890	4793 50 17	4743 52 15	4691 51 15	4640 52 14	4588 52 13	4536 52 12	4484 53 11	4431 53 11	4378 10	890
900	4810 52	4758 52	4706 52	4654 53	4601 53	4548 53	4495 53	4442 54	4388	900

$\phi = 25^\circ$ . TERMINAL VELOCITY.

TABLE VI.

Log. C (c)	1.015 (10.351) dif.	1.035 (10.839) dif.	1.055 (11.350) dif.	1.075 (11.885) dif.	1.095 (12.445) dif.	1.115 (13.032) dif.	1.135 (13.646) dif.	1.155 (14.289) dif.	1.175 (14.962) dif.	Log. C (c)
V.										V.
600	3577 40 17	3617 42 18	3659 44 19	3703 44 21	3747 46 23	3793 47 24	3840 48 25	3888 49 27	3937 28	600
610	3594 41 17	3635 43 18	3678 46 19	3724 46 20	3770 47 21	3817 48 23	3865 50 25	3915 50 26	3965 28	610
620	3611 42 16	3653 44 18	3697 47 20	3744 47 21	3791 49 22	3840 50 23	3890 51 24	3941 52 25	3993 27	620
630	3627 44 17	3671 46 18	3717 48 19	3765 48 20	3813 50 22	3863 51 23	3914 52 24	3966 54 26	4020 27	630
640	3644 45 17	3689 47 19	3736 49 20	3785 50 21	3835 51 22	3886 52 24	3938 54 25	3992 55 26	4047 28	640
650	3661 47 16	3708 48 17	3756 50 19	3806 51 21	3857 53 23	3910 53 24	3963 55 25	4018 57 26	4075 27	650
660	3677 48 16	3725 50 17	3775 52 19	3827 53 20	3880 54 21	3934 54 22	3988 56 24	4044 58 26	4102 27	660
670	3693 49 16	3742 52 17	3794 53 19	3847 54 21	3901 55 23	3956 56 24	4012 58 25	4070 59 26	4129 27	670
680	3709 50 16	3759 54 18	3813 55 18	3868 56 20	3924 56 22	3980 57 24	4037 59 25	4096 60 26	4156 27	680
690	3725 52 16	3777 54 17	3831 57 19	3888 58 20	3946 58 21	4004 58 23	4062 60 25	4122 61 26	4183 27	690
700	3741 53 16	3794 56 17	3850 58 18	3908 59 19	3967 60 20	4027 60 21	4087 61 23	4148 62 24	4210 26	700
710	3757 54 16	3811 57 17	3868 59 18	3927 60 20	3987 61 21	4048 62 22	4110 62 23	4172 64 25	4236 26	710
720	3773 55 15	3828 58 17	3886 61 18	3947 61 19	4008 62 21	4070 63 22	4133 64 23	4197 65 25	4262 26	720
730	3788 57 16	3845 59 17	3904 62 19	3966 63 20	4029 63 21	4092 64 22	4156 66 24	4222 66 25	4288 27	730
740	3804 58 16	3862 61 17	3923 63 18	3986 64 19	4050 64 20	4114 66 22	4180 67 23	4247 68 25	4315 26	740
750	3820 59 16	3879 62 17	3941 64 18	4005 65 19	4070 66 20	4136 67 21	4203 69 23	4272 69 24	4341 26	750
760	3836 60 16	3896 63 17	3959 65 18	4024 66 19	4090 67 21	4157 69 22	4226 70 23	4296 71 25	4367 27	760
770	3852 61 16	3913 64 17	3977 66 18	4043 68 19	4111 68 20	4179 70 22	4249 72 24	4321 73 25	4394 26	770
780	3868 62 16	3930 65 17	3995 67 18	4062 69 19	4131 70 20	4201 72 22	4273 73 23	4346 74 25	4420 26	780
790	3884 63 16	3947 66 17	4013 68 18	4081 70 19	4151 72 20	4223 73 21	4296 75 23	4371 75 25	4446 27	790
800	3900 64 15	3964 67 16	4031 69 17	4100 71 19	4171 73 20	4244 75 21	4319 77 23	4396 77 24	4473 26	800
810	3915 65 16	3980 68 17	4048 71 18	4119 72 19	4191 74 21	4265 77 22	4342 78 23	4420 79 24	4499 26	810
820	3931 66 16	3997 69 17	4066 72 18	4138 74 19	4212 75 20	4287 78 22	4365 79 23	4444 81 24	4525 26	820
830	3947 67 15	4014 70 16	4084 73 18	4157 75 19	4232 77 20	4309 79 21	4388 80 23	4468 83 25	4551 26	830
840	3962 68 16	4030 72 17	4102 74 18	4176 76 19	4252 78 20	4330 81 22	4411 82 23	4493 84 25	4577 26	840
850	3978 69 15	4047 73 17	4120 75 18	4195 77 19	4272 80 20	4352 82 21	4434 84 23	4518 85 25	4603 27	850
860	3993 71 15	4064 74 16	4138 76 17	4214 78 18	4292 81 20	4373 84 22	4457 86 23	4543 87 24	4630 26	860
870	4008 72 15	4080 75 16	4155 77 17	4232 80 19	4312 83 21	4395 85 22	4480 87 23	4567 89 25	4656 26	870
880	4023 73 15	4096 76 16	4172 79 17	4251 82 19	4333 84 20	4417 86 22	4503 89 24	4592 90 25	4682 26	880
890	4038 74 15	4112 77 16	4189 81 18	4270 83 19	4353 86 21	4439 88 22	4527 90 23	4617 91 25	4708 27	890
900	4053 75	4128 79	4207 82	4289 85	4374 87	4461 89	4550 92	4642 93	4735	900
V.										V.
Log. C (c)	(10.351) dif.	(10.839) dif.	(11.350) dif.	(11.885) dif.	(12.445) dif.	(13.032) dif.	(13.646) dif.	(14.289) dif.	(14.962) dif.	Log. C (c)
1.015	1.035	1.055	1.075	1.095	1.115	1.135	1.155	1.175		Log. C

$V = 760 \text{ m/s}$ 

## MAXIMUM ORDINATE.

$\phi$ deg.	Log C C	0.650 4.467	0.700 5.012	0.750 5.623	0.800 6.310	0.850 7.079	0.900 7.943	0.950 8.913	1.000 10.00	1.050 11.22	1.100 12.59	1.150 14.13	1.200 15.85	1.250 17.78
0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
1	8.5	8.5	8.6	8.6	8.6	8.7	8.7	8.7	8.7	8.8	8.8	8.8	8.8	8.8
2	32.2	32.6	32.9	33.2	33.4	33.6	33.8	34.1	34.3	34.5	34.6	34.8	34.8	34.8
3	68.9	70.0	70.9	71.9	72.7	73.6	74.2	74.8	75.3	75.9	76.3	76.6	76.9	76.9
4	117.2	120.1	121.5	124.1	125.2	127.5	128.4	129.4	130.6	132.0	133.3	134.8	135.2	135.2
5	175.7	180.4	183.4	187.7	190.1	194.1	195.8	198.3	200.8	203.9	205.0	207.8	208.5	208.5
6	243.6	251.0	255.7	262.3	266.1	272.1	275.5	280.8	283.6	288.4	290.8	295.0	296.2	296.2
7	320.7	330.4	337.3	346.9	353.2	361.6	366.8	374.6	378.8	385.7	389.0	395.1	398.0	398.0
8	406.4	418.0	428.1	440.9	449.7	461.4	469.0	479.4	486.2	495.2	500.5	508.5	513.4	513.4
9	497.0	513.1	526.9	543.3	555.7	570.6	581.7	595.3	604.6	616.8	624.5	634.4	641.4	641.4
10	593	615	633	654	670	689	704	721	734	749	760	773	782	782
11	697	724	747	773	793	816	835	857	873	892	906	923	936	936
12	807	839	867	898	924	952	975	1001	1022	1045	1064	1082	1100	1100
13	922	960	994	1028	1062	1095	1124	1154	1180	1206	1232	1253	1275	1275
14	1043	1084	1126	1165	1206	1244	1281	1315	1348	1378	1408	1431	1462	1462
15	1168	1215	1265	1310	1358	1400	1445	1484	1525	1560	1596	1627	1659	1659
16	1298	1352	1409	1462	1516	1565	1616	1662	1709	1750	1792	1829	1866	1866
17	1433	1494	1558	1618	1680	1737	1795	1848	1901	1949	1998	2040	2083	2083
18	1573	1640	1713	1780	1850	1915	1979	2041	2100	2156	2211	2260	2309	2309
19	1716	1792	1872	1948	2026	2098	2170	2240	2307	2371	2433	2490	2544	2544
20	1862	1948	2036	2120	2205	2288	2367	2446	2522	2594	2664	2728	2788	2788
21	2013	2107	2203	2298	2390	2484	2572	2659	2743	2825	2901	2973	3040	3040
22	2167	2271	2375	2480	2582	2684	2783	2878	2972	3062	3146	3227	3302	3302
23	2325	2438	2552	2667	2779	2891	3000	3104	3207	3306	3399	3488	3572	3572
24	2486	2610	2734	2857	2980	3102	3221	3336	3448	3558	3659	3757	3849	3849
25	2651	2784	2918	3053	3186	3318	3448	3574	3696	3815	3926	4034	4134	4134
26	2819	2963	3107	3253	3396	3540	3680	3817	3951	4078	4199	4317	4426	4426
27	2991	3145	3299	3457	3611	3766	3916	4065	4210	4347	4480	4603	4725	4725
28	3166	3330	3496	3664	3830	3994	4158	4318	4474	4623	4768	4896	5033	5033
29	3344	3519	3695	3874	4051	4228	4405	4576	4744	4906	5060	5197	5348	5348
30	3524	3710	3898	4089	4278	4467	4657	4838	5018	5193	5354	5506	5671	5671
32	3894	4103	4314	4528	4748	4958	5171	5377	5583	5781	5969	6147	6326	6326
34	4274	4506	4743	4979	5225	5463	5700	5938	6165	6388	6603	6809	7015	7015
36	4665	4919	5181	5446	5712	5981	6245	6510	6767	7013	7253	7488	7725	7725
38	5065	5342	5628	5922	6213	6510	6802	7092	7378	7653	7919	8169	8415	8415
40	5471	5774	6085	6404	6726	7049	7369	7692	8001	8306	8602	8889	9175	9175
42	5883	6213	6551	6896	7247	7597	7949	8299	8637	8968	9293	9609	9925	9925
44	6300	6657	7024	7395	7774	8156	8533	8910	9282	9637	9992	10337	10682	10682
46	6724	7104	7499	7900	8306	8717	9126	9529	9929	10315	10695	11075	11455	11455
48	7148	7556	7976	8407	8842	9280	9720	10154	10579	11000	11415	11825	12235	12235
50	7572	8009	8455	8914	9380	9846	10313	10779	11240	11695	12155	12615	13075	13075
52	7995	8457	8933	9420	9915	10413	10907	11399	11890	12380	12870	13360	13850	13850
54	8414	8905	9410	9923	10446	10972	11499	12016	12533	13050	13567	14084	14601	14601
56	8827	9346	9878	10422	10972	11526	12080	12627	13174	13721	14268	14815	15362	15362
58	9236	9778	10338	10909	11490	12070	12651	13226	13800	14375	14950	15525	16100	16100
60	9632	10202	10786	11386	11992	12603	13208	13808	14408	15008	15608	16208	16808	16808
62	10017	10612	11223	11847	12481	13118	13746	14374	15002	15630	16258	16886	17514	17514
64	10385	11006	11644	12294	12952	13613	14274	14935	15596	16257	16918	17579	18240	18240
66	10740	11383	12044	12718	13401	14086	14771	15456	16141	16826	17511	18196	18881	18881
68	11073	11739	12424	13121	13827	14535	15243	15951	16659	17367	18075	18783	19491	19491
70	11388	12073	12779	13498	14224	14954	15684	16414	17144	17874	18604	19334	20064	20064
72	11678	12384	13108	13847	14594	15341	16088	16835	17582	18329	19076	19823	20570	20570
74	11943	12667	13409	14166	14931	15698	16465	17232	18000	18767	19534	20301	21068	21068
76	12181	12920	13681	14454	15234	16015	16800	17585	18370	19155	19940	20725	21510	21510
78	12392	13146	13920	14708	15502	16298	17100	17900	18700	19500	20300	21100	21900	21900
80	12572	13338	14123	14925	15732	16545	17365	18185	19005	19825	20645	21465	22285	22285
82	12722	13498	14294	15104	15921	16745	17575	18410	19245	20080	20915	21750	22585	22585
84	12839	13623	14427	15246	16070	16900	17735	18575	19415	20255	21095	21935	22775	22775
86	12924	13714	14522	15346	16176	17015	17860	18710	19560	20410	21260	22110	22960	22960
88	12976	13768	14580	15408	16240	17075	17920	18770	19620	20470	21320	22170	23020	23020
90	12993	13786	14596	15432	16265	17100	17945	18795	19645	20495	21345	22195	23045	23045

## MAXIMUM ORDINATE.

 $V = 800 \text{ m/s}$ 

$\phi$ deg.	Log C	0.650	0.700	0.750	0.800	0.850	0.900	0.950	1.000	1.050	1.100	1.150	1.200	1.250
C	C	4.467	5.012	5.623	6.310	7.079	7.943	8.913	10.00	11.22	12.59	14.13	15.85	17.78
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	9.3	9.4	9.5	9.5	9.6	9.6	9.6	9.7	9.7	9.7	9.7	9.7	9.7	9.8
2	35.2	35.8	36.1	36.7	36.8	37.2	37.3	37.8	37.7	38.0	38.0	38.4	38.3	38.3
3	75.7	77.1	77.8	79.3	80.0	81.2	81.6	82.7	83.0	83.9	84.0	85.0	85.2	85.2
4	128.1	131.3	133.0	136.1	137.5	140.2	141.2	143.4	144.1	146.3	146.8	148.6	149.3	149.3
5	191.7	197.2	200.6	205.5	208.3	212.8	215.0	219.0	220.7	224.0	225.5	228.6	229.7	229.7
6	265.1	273.6	278.9	286.6	291.2	298.1	302.0	308.1	311.5	316.7	319.7	324.0	326.3	326.3
7	347.6	359.3	367.5	378.3	385.5	395.2	401.6	410.4	415.7	423.6	428.0	434.3	438.0	438.0
8	438.0	453.9	465.5	480.3	490.4	503.6	512.9	524.9	532.7	543.4	550.1	558.7	564.5	564.5
9	535.9	556.6	572.1	591.1	605.2	622.5	635.3	651.1	661.9	675.8	685.4	697.2	705.3	705.3
10	641	667	687	711	729	751	768	788	802	820	833	849	859	859
11	753	784	809	838	862	888	910	934	954	976	993	1011	1027	1027
12	871	907	939	974	1003	1034	1062	1090	1117	1142	1164	1186	1207	1207
13	994	1036	1075	1116	1152	1187	1223	1255	1289	1318	1347	1372	1398	1398
14	1123	1170	1218	1264	1308	1350	1393	1431	1471	1505	1541	1571	1602	1602
15	1257	1310	1367	1418	1472	1521	1571	1616	1662	1703	1744	1781	1816	1816
16	1396	1457	1521	1580	1641	1698	1756	1809	1862	1910	1957	2000	2042	2042
17	1540	1609	1681	1748	1810	1883	1948	2010	2069	2126	2180	2230	2278	2278
18	1688	1765	1847	1923	2003	2075	2148	2218	2286	2351	2412	2470	2525	2525
19	1841	1927	2017	2103	2191	2274	2355	2435	2511	2585	2654	2720	2782	2782
20	1998	2094	2191	2288	2384	2479	2569	2659	2744	2827	2905	2980	3049	3049
21	2159	2264	2369	2479	2584	2689	2790	2890	2985	3077	3165	3248	3326	3326
22	2323	2439	2554	2675	2790	2906	3018	3127	3233	3335	3432	3524	3612	3612
23	2490	2617	2745	2874	3001	3128	3251	3372	3488	3601	3706	3810	3906	3906
24	2662	2801	2941	3077	3217	3356	3491	3622	3750	3873	3988	4103	4209	4209
25	2838	2987	3139	3287	3439	3589	3735	3878	4018	4153	4280	4404	4521	4521
26	3017	3178	3341	3503	3666	3828	3985	4141	4294	4439	4581	4712	4841	4841
27	3200	3372	3546	3722	4898	4072	4242	4410	4575	4732	4887	5029	5169	5169
28	3386	3570	3755	3945	4134	4319	4505	4685	4862	5032	5198	5354	5504	5504
29	3575	3771	3968	4171	4372	4572	4771	4964	5155	5338	5516	5686	5846	5846
30	3767	3976	4186	4401	4616	4830	5042	5249	5455	5651	5841	6024	6191	6191
32	4161	4394	4632	4872	5116	5359	5599	5838	6067	6294	6510	6722	6929	6929
34	4567	4824	5090	5359	5632	5903	6172	6442	6702	6956	7201	7434	7657	7657
36	4982	5265	5559	5859	6162	6463	6764	7062	7355	7639	7913	8177	8431	8431
38	5405	5719	6039	6368	6701	7035	7370	7699	8022	8338	8642	8937	9221	9221
40	5837	6180	6529	6888	7251	7620	7987	8350	8702	9052	9393	9724	10045	10045
42	6276	6647	7028	7417	7812	8214	8614	9010	9395	9776	10155	10524	10883	10883
44	6722	7120	7534	7954	8384	8817	9250	9680	10100	10510	10916	11317	11713	11713
46	7172	7600	8043	8498	8960	9425	9892	10355	10812	11260	11699	12138	12567	12567
48	7624	8084	8557	9043	9539	10038	10537	11033	11524	12010	12491	12968	13441	13441
50	8076	8567	9073	9590	10118	10653	11184	11714	12240	12761	13278	13791	14299	14299
52	8527	9049	9587	10137	10697	11265	11830	12393	12951	13505	14055	14601	15143	15143
54	8976	9527	10096	10681	11274	11874	12473	13068	13658	14243	14824	15401	15974	15974
56	9419	10000	10601	11218	11845	12476	13108	13732	14351	14966	15577	16184	16787	16787
58	9853	10466	11098	11745	12404	13068	13732	14391	15046	15697	16344	16987	17626	17626
60	10277	10920	11583	12261	12950	13647	14341	15041	15736	16427	17114	17797	18476	18476
62	10688	11360	12053	12762	13483	14208	14937	15671	16410	17154	17893	18628	19359	19359
64	11086	11784	12505	13244	13994	14750	15511	16277	17048	17824	18595	19362	20125	20125
66	11466	12191	12938	13704	14483	15265	16051	16841	17634	18431	19224	20013	20798	20798
68	11825	12576	13349	14140	14945	15753	16564	17379	18197	19019	19846	20669	21488	21488
70	12160	12935	13734	14550	15378	16210	17046	17886	18729	19576	20427	21273	22114	22114
72	12472	13269	14091	14930	15782	16638	17497	18359	19224	20093	20966	21834	22697	22697
74	12757	13574	14416	15277	16150	17026	17905	18786	19669	20556	21447	22342	23231	23231
76	13013	13848	14709	15590	16479	17371	18264	19159	20056	20956	21859	22766	23667	23667
78	13239	14091	14968	15864	16771	17680	18591	19503	20416	21331	22249	23170	24085	24085
80	13434	14299	15190	16100	17020	17941	18863	19786	20711	21638	22568	23499	24423	24423
82	13596	14472	15373	16294	17226	18159	19093	20028	20964	21901	22840	23771	24694	24694
84	13722	14608	15517	16446	17390	18335	19280	20226	21173	22121	23070	24020	24962	24962
86	13814	14705	15622	16556	17505	18455	19405	20356	21308	22261	23215	24170	25117	25117
88	13869	14763	15685	16625	17576	18527	19478	20430	21383	22337	23292	24248	25195	25195
90	13889	14779	15708	16650	17603	18556	19509	20463	21418	22374	23331	24288	25236	25236

THE DETERMINATION OF INITIAL VELOCITY AND COEFFICIENT OF FORM BY EXPERIMENTAL FIRING AT THE PROVING GROUND. THE DETERMINATION OF RANGE-TABLE VALUES OF RANGE, ANGLE OF DEPARTURE, ANGLE OF FALL, TIME OF FLIGHT, STRIKING VELOCITY, AND MAXIMUM ORDINATE (RANGE-TABLE COLUMNS 1, 2, 3, 4, 5, and 8).

801. Such tables as the A.L.V.F. Tables and War Department Tables are called *ballistic tables*, and they constitute a basis for the solution of trajectories corresponding to any combination of  $V$  and  $C$  that is likely to occur in practice.

The distinction between ballistic tables and range tables

In order that the data for a given gun may be available in convenient form, a *range table* is prepared for each gun. A range table is, first of all, a tabulation of elements of trajectories corresponding to the initial velocity and projectile of a particular gun, as compared to a ballistic table, which is applicable to many velocities and many projectiles. In this sense a ballistic table may be thought of as a master range table.\* A range table has the additional feature of making available the data corresponding to the particular gun sufficiently in detail to render interpolation unnecessary in connection with ordinary service problems.

Although a range table is designed to serve for a particular gun having a specified initial velocity and projectile, it is necessary to provide for normal variations in these factors, as well as in others that affect the trajectory. In order to avoid the complications that would result from adopting a multiple-entry system to take care of such variations, it is the practice to tabulate in a range table the values of the desired elements under a standard set of conditions, and to provide therein also additional data by means of which the tabulated values may be adjusted for variations from the assumed standards. The standard conditions assumed for range-table values are,

Standard conditions assumed for range-tables

- (a) That the projectile leaves the gun with the designed velocity.
- (b) That the projectile has the designed weight and form.
- (c) That the air at the surface is at the standard temperature (59°F.), pressure (29.53 inches of mercury), and saturation (78%), and that its variations aloft follow the assumed standard relations (art. 419-420).
- (d) That there is no wind.
- (e) That the gun is motionless.
- (f) That the target is motionless. (Although motion of the target does not affect the trajectory itself, it enters into the problem of determining the trajectory that will reach the desired objective.)
- (g) That the earth is motionless.
- (h) That the gun and target are in the same horizontal plane.
- (i) That the gun is warm.

\* This is particularly true in the case of tables such as the A.L.V.F. Tables and War Department Tables, from which the desired elements are obtained directly. Some other ballistic tables, such as Ingalls' Tables (art. 511 and Appendix A), list not the elements themselves, but functions which are required for the solution of formulas for the elements.

802. We shall deal, in this chapter, with the problem of determining the values of range, angle of fall, time of flight, striking velocity, and maximum ordinate, corresponding to various angles of departure, that are tabulated in Columns 1, 2, 3, 4, 5, and 8 of the range table. All of these are elements for which values may be obtained by the processes of solution already outlined in foregoing chapters. Our immediate problem in the present chapter has to do with the application of the processes previously considered to the determination of values of these elements under the standard conditions assumed for a range table.

It is to be noted that the standard conditions stated above are in accord with the assumptions which were defined originally in article 213, and that no departure from the latter assumptions has been made in connection with the processes of solution and preparation of the ballistic tables which have been discussed

The applicability of ballistic tables to the determination of range-table values

in foregoing chapters. These ballistic tables therefore will yield directly values that correspond to the standards assumed for range tables. Insofar as the argument  $V$  is concerned, it is merely necessary to enter with the chosen standard value of  $V$ .

The determination of the argument  $C$  also is very simple, insofar as the factors  $\delta$ ,  $w$ , and  $d$  are concerned, for  $\delta$  equals unity, and  $w$  and  $d$  are simply the designed weight and diameter of the projectile to which the range table is to apply. The remaining factor,  $i$ , which is contained in  $C$ , cannot be assigned a standard value, however, and must be evaluated by experimental firing.

The numerous considerations that enter into the value of the coefficient of form,  $i$ , have been discussed in detail in articles 413-417 and 624-626. All that has previously been said with respect to the value of  $i$  may be summarized by the statement that *the value assigned to  $i$  must be such as to cause agreement between the*

General principle involved in determination of  $i$

*values of elements of the trajectory as determined by computation and as determined by experiment.* Our problem thus requires, first of all, measurement of the range  $X$  which results from a firing of the given gun and projectile, and of the values of  $\phi$ ,  $V$ ,  $\delta$ ,  $w$ , and  $d$  for that firing,—and determination of the value of  $i$  which, in combination

with the measured  $\phi$ ,  $V$ ,  $\delta$ , and  $w$ , and  $d$ , reproduces the measured  $X$  by whatever system of computation or tables we have chosen to use. Since the use of ballistic tables is an essential expedient in carrying through this process, as already noted in article 624, it is important that due consideration be given to all variations between the conditions under which the experimental firing is conducted and the standard conditions upon which the ballistic tables are based. We shall now examine briefly the principal features that enter into the experimental ranging of a gun and into the determination of the measured values of  $\phi$ ,  $V$ ,  $\delta$ ,  $w$ ,  $d$ , and  $X$  that are applicable to the ballistic tables; we may proceed thence to our more immediate problem of using these values to find  $i$  and of using the latter in the construction of the range table.

General features of experimental ranging at the proving ground

803. The experimental ranging of a gun at the proving ground, for the determination of range-table data, consists of the firing of several shots (from four to seven) at each of several angles of departure ( $8^\circ$ ,  $15^\circ$ ,  $25^\circ$ ,  $35^\circ$ ,  $45^\circ$ ,  $15^\circ$ , is a typical series), the gun being mounted in its regular service mount.

The following observations are made in connection with the firing.

- (a) The gun is laid by quadrant at exactly the required angle of departure before each shot.
- (b) The initial velocity for each shot fired at an angle of departure of  $15^\circ$  or less is determined from the velocity measured by oscillograph at a

short distance from the muzzle, and the initial velocities of the several shots fired at each of these angles of departure are averaged to give the initial velocities for these groups of shots. Since the rigging of screens or coils for velocity measurement is impracticable for angles of departure above about  $15^\circ$ , the initial velocities for the groups fired above  $15^\circ$  are determined by interpolation between the initial velocities actually measured for two groups at  $15^\circ$ , one preceding and one following the groups fired above  $15^\circ$ .\* Due care having been taken to preserve uniformity as to weight and temperature of the powder charge throughout the firing, comparison of the measured velocities of the two  $15^\circ$  groups affords a good estimate of the loss of initial velocity due to erosion, in terms of rounds fired, which is the basis of the interpolation for the initial velocities of the groups fired above  $15^\circ$ .

- (c) Throughout the firing meteorological observations are made, both at the surface and aloft, for the determination of the atmospheric density factor and wind. For each group of shots fired at the same angle of departure, an average ballistic density and wind are determined from the observations taken during the firing of that group.
- (d) The point of fall of each shot is plotted by intersecting bearings taken at four observation stations. The distance from the gun to the point of fall is the uncorrected observed range. The perpendicular distance from the line of fire to the point of fall is the uncorrected observed lateral deviation. For each group of shots fired at the same angle of departure, average values of the uncorrected observed range and lateral deviation are determined from the individual values observed for the shots of that group. For use in connection with reduction of the observed range to the standard datum plane, the average height of the gun above tide level is recorded for each group. Also, the average weight of projectile for each group is recorded.
- (e) Any shots fired while the gun is cold are excluded from the ranging data. Some of the features outlined above will now be considered in somewhat greater detail.

804. It is to be noted that in laying the gun for the experimental ranging no allowance is made for the angle of jump. The angle of departure actually obtained therefore varies from the angle indicated by the quadrant by the amount of the angle of jump, and the observed range corresponds to the quadrant elevation plus or minus the angle of jump rather than to the quadrant elevation as actually laid. However, the angle of jump is an inherent part of the angle of departure and will be present in any subsequent firing of the gun as well as in the experimental range firing. Nothing is to be gained by attempting to separate the angle of jump from the angle of departure at the experimental ranging, as will appear from the following example. Let us suppose that a gun has been laid at the quadrant elevation  $15^\circ$  for experimental ranging, and that the angle of jump is known to be  $+2'$ ; also that the resulting range, reduced to range-table standards, has been found to be 20,000 yards. The range 20,000 yards therefore corresponds actually to the angle of departure  $15^\circ 02'$ ; if we should tabulate the latter value in the range table, we would then be obliged to tabulate also the angle of jump  $+2'$ , and to subtract this from  $15^\circ 02'$  to obtain the  $15^\circ$  which we already know is required to

\* It is expected that facilities for measuring velocities for shots fired at elevations up to at least  $30^\circ$  will become available in the near future.

give the range 20,000 yards. The identical result is obtained by tabulating  $15^\circ$  against 20,000 yards in the first place. The same applies in any case, since however we may determine the angle of jump and whatever we may find its value to be at the experimental ranging, we can do no better than to assume that this value will remain the same for the subsequent service firing of the gun at the same elevation.\*

This method of avoiding separate consideration of the angle of jump obviously is a source of disagreement between computed and observed results, for it results in comparison of the observed range for one angle of departure with the computed range for a slightly different angle of departure. However, the angle of jump is small enough in any normal case to make this a feature of little importance even for theoretical considerations, and of no consequence at all in connection with our practical problem of constructing a range table. It is sufficient to note that the small discrepancies between observed and theoretical results that are attributable to this procedure are, like many others, accounted for by the coefficient of form.

805. What has been said above in connection with jump applies generally also to droop. Although the droop itself can be measured, its effect on the angle of departure cannot very well be separated from that due to the jump, for the droop may operate not only to cause the angle of departure to be modified slightly by the curvature of the bore, but also to cause a whip of the bore as the projectile travels through it. Whatever pains might be taken to evaluate these effects separately at the experimental ranging, we would still be obliged to assume identical effects for the subsequent service firings. The elimination of erratic or excessive droop or jump is a requirement of acceptable design of guns and their mounts.

It is to be noted, however, that in the service use of a gun the orientation of the sights by bore-sighting results in the use of the chord joining the centers of the breech and muzzle as the axis of the bore, whereas the laying of the gun by means of a quadrant, as is done at the proving ground, results in the use of the designed axis of the bore. A gun laid level by means of the axis of boresighting therefore actually has a slight elevation as compared to the same gun laid level by quadrant, the difference being caused by the droop (which lowers the muzzle end of the axis of boresighting). The effect of this is to cause guns in actual service to overshoot their range tables according to the amount of droop that existed at the time of boresighting. The deviation of the chord joining the centers of the breech and muzzle from the designed axis of the bore normally does not exceed a few minutes of arc.

806. The determination of initial velocity involves two principal features, viz.,

- |   |   |
|---|---|
| Experimental<br>determination<br>of initial<br>velocity | (a) The measurement of the remaining velocity at a short distance from the muzzle.  |
|   | (b) The calculation of the initial velocity which corresponds to this remaining velocity, considering the velocity reduction that has occurred during the flight of the projectile from the muzzle to the point of measurement. |

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\* The wording in the heading of Column 2 of our range tables is misleading in this connection, if interpreted literally. It is intended to be implied that the angle of jump has been considered in the tabulated values of angle of departure in the manner described above, and that no further corrections for jump are applicable. The correct interpretation is that the range tabulated in Column 1 corresponds to the angle of elevation tabulated in Column 2 plus the normal jump at that elevation.

The principles involved in the first of these features have already been discussed in article 403. The present practice at the Naval Proving Ground is to use an oscillograph for this purpose. The coils are so rigged that their heights may be adjusted for firing at angles of departure up to about  $15^\circ$ . Illustrations of the Naval Proving Ground layout for the measurement of velocities, and detailed descriptions of the apparatus used, are given in articles 773–781, *Naval Ordnance, 1933*.

The second of the above features involves, in effect, the solution of the trajectory between the muzzle and the point to which the measured remaining velocity pertains, which may be taken to be the point midway between the two coils. We have the values of  $x$ ,  $y$ , and  $v$  for this point in the trajectory. Although the exact value of  $\theta$  for this point is not directly available, no appreciable error will result from assuming that the short portion of the trajectory involved is a straight line, and that  $\theta = \phi$ . Knowing  $x$ ,  $y$ ,  $v$ , and  $\theta$  for the given point, we can now work back to the origin and thus determine  $V$ . This solution might be made by numerical integration as outlined in Chapter 6, but for the conditions of the present problem the application of Siacci's Method affords a much simpler solution with negligible sacrifice of accuracy, and the latter is actually used.

807. The following formula, which has been derived in article 510, is directly applicable to the present problem.

$$x = C_s(S_u - S_v). \quad (519)$$

Calculation  
of initial  
velocity by  
Siacci's  
Method

According to this formula,  $x$  is the horizontal component of the distance covered by the projectile from the origin, where its velocity is  $V$ , to the point where its pseudo velocity is  $u$ ;  $S_v$  and  $S_u$  are the values of Siacci's space function for these velocities, and may be found from tables (art. 511). From formula (507) we have the relation

$$u = v \cos \theta \sec \phi \quad (801)$$

and since  $\theta$  in this case varies but slightly from  $\phi$ , it is sufficiently accurate for practical purposes in this connection to assume  $u = v$ , that is, that the pseudo velocity at the point of measurement equals the measured remaining velocity at that point. Transposing (519) we have

$$S_v = S_u - \frac{x}{C_s} \quad (802)$$

which may be solved by substituting for  $S_u$  the tabulated value of  $S$  for the velocity  $u = v$ , for  $x$  the horizontal component of the distance from the gun to the point midway between the oscillograph coils, and for  $C_s$  its value as given by formula (513).\* The required value of  $V$  is then that which corresponds to the tabulated value of  $S_v$  found in this manner.†

Since the ballistic coefficient enters into the above determination, it is necessary to have a value of  $i$  before the initial velocity can be determined, although the initial velocity so determined is to enter eventually into the determination of  $i$  itself. Ordinarily a sufficiently accurate estimate of  $i$  for use in the determination of initial velocity is available from previous work. If the value of  $i$  eventually found for a given trajectory should vary materially from that assumed for the determination of

\* In actual practice,  $C$  is generally used in place of  $C_s$ ,  $f_a$  and  $\beta$  being disregarded in view of their slight effect on the determination of  $V$ .

† These tables are given in the 1926 and 1930 editions of *Range and Ballistic Tables* but are omitted from the present edition due to the very limited references made to them in the present textbook.

the initial velocity used for that trajectory, the entire computation may be repeated, starting with a new determination of the initial velocity according to the better value of  $i$  established by the results of the first approximation. Similarly, if no previous knowledge of the required value of  $i$  should be available, a solution can be made by assuming an estimated value and making successive approximations until agreement is established within the desired limits of accuracy. This involves but little difficulty, considering the relatively small effect that variations in  $i$  have in the determination of the initial velocity.

**808.** It is worthy of note that the process outlined above, or any equivalent computational process for deducing the initial velocity from a velocity measured at a distance from the muzzle, does not in fact yield the true value of the projectile's velocity as it leaves the muzzle, but yields instead an initial velocity under the assumption that the projectile has suffered normal retardation from the instant of projection. Actually the acceleration of the projectile continues during a flight of as much as fifty yards from the muzzle, due to the action of the blast which escapes from the gun at high velocity and surrounds the projectile during the initial stage of its flight. The actual velocity at the muzzle therefore is not the greatest velocity attained by the projectile, nor is it the initial velocity in the sense in which the latter is employed in exterior ballistics.

The process used determines neither the true velocity at the muzzle nor the true maximum velocity actually attained by the projectile. The value of  $V$  so determined is a fictitious initial velocity which, according to the assumed velocity-retardation relation and for the actual distance covered by the projectile up to the point of measurement, satisfies the actual remaining velocity at that point. The only appreciable inaccuracy that results from this manner of handling the problem applies to values of the velocity itself for points short of the point of measurement. However, this is of no consequence in connection with the trajectory computation as a whole, since the initial velocity determined in the manner described has been chosen so as to cause agreement between computed and observed values at the point of measurement; in other words, it is the equivalent of the actual velocities between the muzzle and point of measurement. On the other hand, if means should be devised for obtaining an accurate measure of the velocity of the projectile at the muzzle, it nevertheless would remain necessary to make measurements also beyond the muzzle, at least up to the point where the projectile clears the blast from the gun and begins to encounter normal retardation. We would then have the choice of making a separate computation for the portion of the trajectory up to this point, or of deducing an equivalent initial velocity applicable to the usual system of computation. The latter is, in effect, what is accomplished by making a single measurement clear of the blast in the first place, and nothing is to be gained by attempting to make a close study of the velocity in the immediate vicinity of the muzzle, as far as our present problem is concerned.

**809.** The meteorological observations taken during the experimental ranging furnish data both for determining the value of  $\delta$  that is to be used in connection with the other measured values from which  $i$  is to be deduced, and for determining corrections required to adjust the observed range and lateral deviation to the range-table condition of no wind. Accurate determination of air density and wind conditions encountered by the projectile throughout its flight requires measurements at several levels of altitude up to the height of the maximum ordinate. Aloft densities are determined

Nature of the experimentally determined value of initial velocity

Corrections for actual air density and wind

from measurements of air temperature and barometric pressure made by aircraft at approximately each 1000 feet of altitude, and are used for determining the ballistic density. Or, if aloft observations are not available, the ballistic density is determined from Table IV, as outlined in article 424. Aloft winds are measured by plotting the path of a small pilot balloon having a known ascensional rate, and are used for determining the ballistic wind.\* (The method of finding the ballistic wind from aloft observations will be gone into further in articles 1114–1115).

810. In order to reduce the observed range to the conditions on which ballistic tables and range tables are based, it is necessary to apply to it corrections for the wind in the line of fire, and for the condition that the observed point of fall is not in the horizontal plane through the gun. The method of determining the correction for wind can be explained to better advantage after we have progressed further with other details of building up the range table, and it will be dealt with in Chapter 11; for the present we may accept that the necessary correction is available.

Two corrections are necessary to refer the observed point of fall to the horizontal plane through the gun, viz.,

- (a) A correction for the height of the gun above the surface of the earth (the tide level in this case).
- (b) A correction for the curvature of the earth.

Although these two corrections are due to entirely different causes, they are essentially similar in nature, as will be apparent from a study of Figure 14. In this figure the gun is assumed to be at  $O$ , at a height  $h_1$  above the surface of the earth.  $OH$  is the horizontal plane through the gun,  $O'H'$  is the horizontal plane

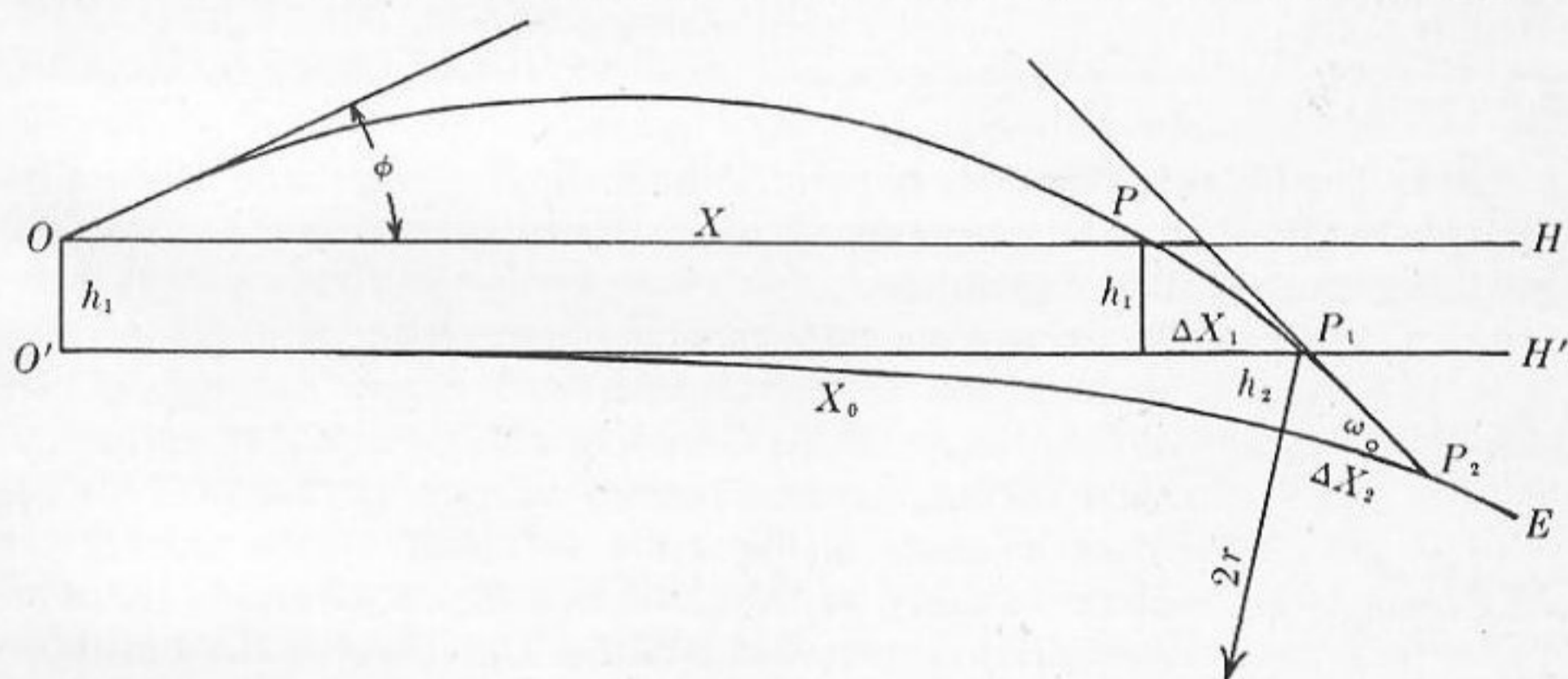


FIGURE 14

tangent to the earth at the point vertically beneath the gun, and  $O'E$  is the curved surface of the earth. The observed point of fall is on the surface of the earth at  $P_2$ , and  $X_0 = O'P_2$  is the observed range. The required range  $X$  is the range to the point where the trajectory again intersects the horizontal plane through the gun, and it is equal to  $OP$ . It is sufficiently correct for practical purposes to assume that the observed range  $X_0 = O'P_2$  exceeds the required range  $X = OP$  by the sum of the two increments  $\Delta X_1$  and  $\Delta X_2$ , and that the latter are equal, respectively, to  $h_1 \cot \omega_0$  and  $h_2 \cot \omega_0$ ,  $\omega_0$  being the angle of fall for the observed range. Although the angles of fall at  $P_1$  and  $P_2$  differ slightly, the difference is small enough

\* The method of measuring aloft winds by means of pilot balloons is described in detail in *Technical Regulations No. 1236-1* (U. S. War Department, 1934).

to be disregarded for the present purpose; a first approximation of the angle of fall based on the uncorrected observed range is sufficiently accurate for the determination of both of these corrections.

For the determination of  $\Delta X_1$ , the value  $h_1$  is available from the records taken at the time of firing, as noted in article 803 (d). The value of  $h_2$  which is required for the determination of  $\Delta X_2$  can be found by applying a theorem of geometry, viz., that the length of the tangent to a circle from an exterior point is the mean proportional between the whole length of a secant from this point, and its external segment. In this case the length of the tangent is  $O'P_1$ , the whole length of the secant is  $2r+h_2$  ( $r$  being the radius of the earth), and  $h_2$  is the external segment. Since  $h_2$  is insignificant in comparison with  $2r$ , we may use  $2r$  as the value of the whole secant; also, no material error will result from using the uncorrected observed range  $X_0$  in place of  $O'P_1$ . Hence we have

$$\frac{h_2}{X_0} = \frac{X_0}{2r}$$

and

$$h_2 = \frac{X_0^2}{2r} = [2.85590 - 10]X_0^2$$

the number in brackets being the colog of the value of  $2r$  in yards. The formulas for the corrections to be applied to the observed range for height of gun and curvature of the earth therefore are, respectively,

Corrections for height of gun and curvature of the earth	$\Delta X_1 = h_1 \cot \omega_0$	(803)
	$\Delta X_2 = h_2 \cot \omega_0 = [2.85590 - 10]X_0^2 \cot \omega_0.$	(804)

811. The following example illustrates the method of computing the corrections for height of gun and curvature of the earth, and gives, also, an idea of the usual magnitude of these corrections.

*Given:* In the experimental ranging of a certain gun at the proving ground, the average uncorrected observed range of five shots fired at angle of departure  $15^\circ$  was 22,842 yards, and their average measured initial velocity was 2608 f.s. For the firing at angle of departure  $40^\circ$ , the average uncorrected observed range for five shots was 38,088 yards and the average initial velocity was 2597 f.s. The height of the gun above tide level at the time of firing was 24 feet in both cases.

Example of  
reduction of  
observed  
range

*Find:* The corrections to the observed range due to height of gun and curvature of the earth for each of the given groups, and the corrected observed ranges (assuming a wind correction of zero yards in each case).

Converting the given ranges and velocities into metric units and then entering Table VI with them, we can find the value of  $\text{Log } C$  in each case and then use the latter to find  $\omega$  for each case; this gives the following results.

<u>For <math>\phi = 15^\circ</math></u>	<u>For <math>\phi = 40^\circ</math></u>
$X_0 = 22,842 \text{ yds.} = 20,887 \text{ m.}$	$X_0 = 38,088 \text{ yds.} = 34,828 \text{ m.}$
$V_0 = 2608 \text{ f.s.} = 794.9 \text{ m.s.}$	$V_0 = 2597 \text{ f.s.} = 791.6 \text{ m.s.}$
$\text{Log } C = 1.11646$	$\text{Log } C = 1.11590$
$\omega_0 = 21^\circ 45'$	$\omega_0 = 52^\circ 00'$

The value  $h_1 = 24$  feet = 8 yards applies to both angles of departure. Hence we have all values required for the solution of (803) and (804), as follows.

<u>For <math>\phi = 15^\circ</math></u>	<u>For <math>\phi = 40^\circ</math></u>
$h_1 = 8$ yds. . . . . log 0.90309	$h_1 = 8$ yds. . . . . log 0.90309
$\omega_0 = 21^\circ 45'$ . . . . . <u>l cot 0.39907</u>	$\omega_0 = 52^\circ 00'$ . . . . . <u>l cot 9.89281 - 10</u>
$\Delta X_1 = 20$ yds. . . . . log 1.30216	$\Delta X_1 = 6$ yds. . . . . log 0.79590
$X_0 = 22,842$ yds. . . . . 2 log 8.71748	$X_0 = 38,088$ yds. . . . . 2 log 9.16158
$\omega_0 = 21^\circ 45'$ . . . . . l cot 0.39907	$\omega_0 = 52^\circ 00'$ . . . . . l cot 9.89281 - 10
$2r =$ . . . . . <u>colog 2.85590 - 10</u>	$2r =$ . . . . . <u>colog 2.85590 - 10</u>
$\Delta X_2 = 94$ yds. . . . . log 1.97245	$\Delta X_2 = 81$ yds. . . . . log 1.91029
$X = 22,842 - 20 - 94 = \underline{22,728}$ yds.	$X = 38,088 - 6 - 81 = \underline{38,001}$ yds.

812. It will be noted that the height of gun correction is greatest at short ranges, and becomes very small at long ranges due to the decrease in  $\cot \omega$ . In the correction for curvature of the earth, the increases of  $h_2$  and decreases of  $\cot \omega$ , as the range increases, partially offset each other, and this correction remains within fairly narrow limits for practically all trajectories (for a given gun and initial velocity) up to about  $40^\circ$ , above which it decreases rapidly. It is of interest to note also the degree of error that results from the assumptions made in the foregoing article, viz., that for determining  $\Delta X_1$  the  $\omega$  for point  $P_1$  is the same as the  $\omega$  at  $P_2$ ; and that for determining  $\Delta X_2$  the uncorrected range  $X_0$  may be used in place of  $O'P_1$  (see Fig. 14). A second approximation, based on the determination of  $P_1$  which is available from the first approximation made above, alters the final result only 0.6 yard for the  $15^\circ$  group and 0.3 yard for the  $40^\circ$  group. It is well to bear in mind that such devices as these often result in material simplification of a problem with but inconsequential sacrifice of accuracy. Thus the uncorrected observed range may be used also for finding the maximum ordinate which is required for determining the ballistic density and ballistic wind pertaining to the experimental ranging, and for determining the corrections to the observed range and lateral deviation due to wind, without causing material inaccuracy.

813. A question may be raised at this point as to the necessity for applying a correction for curvature of the earth to the observed range at the experimental ranging. It appears on first thought that the application of a correction for curvature of the earth at the experimental ranging serves merely to make it necessary to apply a similar correction in the subsequent service use of the gun, and that the latter might be avoided by omitting this correction in the experimental ranging in the first place. This would be true if the gun were to be laid in service firing by the same method that is employed at the proving ground, but this is not the case. In firing from a ship, the gun is laid in elevation with respect to the line of sight, and not by quadrant as at the proving ground. The effect of this difference in the manner of laying the gun will appear from a study of Figure 15.

In Figure 15 the gun is at  $O$ , at a height  $h$  above the water (as on the deck of a ship).  $OH$  is the horizontal plane through the gun, and  $O'E$  is the surface of the water, on which a target is located at  $T$ .  $OT$  is the line of sight, and the gun sights are set so as to cause the gun to be elevated at the angle of elevation  $\phi'$  which determines the trajectory that will intersect the line of sight at  $T$ , i.e., that will give the range  $OT$  measured in the line of sight. According to the as-

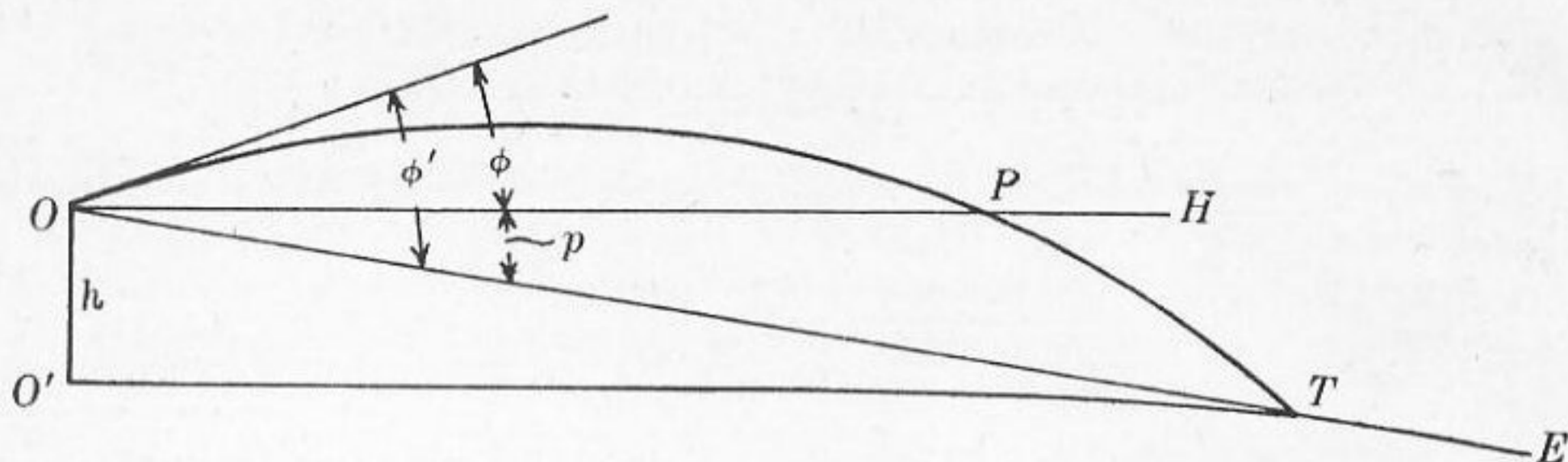


FIGURE 15

sumption of rigidity of the trajectory, as defined in article 319, it now appears that the situation here is the equivalent of having the gun and target in the same horizontal plane, and of using as the angle of elevation  $\phi'$  the angle of departure  $\phi$  tabulated in the range table for the given distance  $OT$ . This being the case, it is evident that neither the height of the gun nor the curvature of the earth enters into the situation, except insofar as they introduce a small angle of position. As shown in the discussion of the assumption of rigidity (arts. 317-319), the effect of such small angles of position as occur in practice is insignificant.\* The assumed range-table condition of a horizontal plane containing both the gun and target is therefore directly applicable to surface fire when the gun is aimed at the target by means of sights.

A different situation obtains when indirect fire is used, for in this case gun elevation is laid with respect to the horizontal plane, as established by an instrument such as the stable-zenith director. Corrections for height of gun and curvature of the earth therefore are applicable in indirect fire, but the conditions under which indirect fire is used in naval practice ordinarily are such as to render corrections of this character of small importance in comparison with other problems that enter into this type of fire.

814. Having examined the methods by means of which the results of actual firings are observed at the experimental ranging conducted at the proving ground, we may now proceed to examine the problem of using this information for the construction of a range table. First of all we must find the value of  $i$  from the results of the experimental ranging, which is done as illustrated in the following problem.

*Given:* In the experimental ranging of the 16''2600 f.s. gun the following average results were obtained for a group of five shots fired at angle of departure  $25^\circ$ : uncorrected observed range, 31,845 yards; observed initial velocity, 2595 f.s.; observed weight of projectiles, 2099 lbs.; observed ballistic density, .961. The corrections to be applied to the observed range were found to be: for ballistic wind (-) 126 yards; for height of gun, (-) 10 yards; for curvature of the earth (-) 95 yards.

*Find:* The coefficient of form that corresponds to the above data.

Example of  
experimental  
determination  
of  $i$

\* Since land artillery is usually laid in elevation by quadrant or the equivalent thereof, the range tables of the U. S. Army are based on the curved surface of the earth rather than on the horizontal datum plane. This is accomplished by allowing the effect of the earth's curvature to remain in the ranges observed at the experimental ranging for range-table data.

The corrected observed range is

$$X = 31,845 - 126 - 10 - 95 = 31,614 \text{ yards.}$$

The corrected observed range and the observed initial velocity, converted to metric units, are

$X = 31,614 \text{ yds.} \dots \log 4.49988$	$V = 2595 \text{ f.s.} \dots \log 3.41414$
$(\text{art. 705}) \dots \log 9.96114 - 10$	$\log 9.48402 - 10$
<hr/> $X = 28,908 \text{ m.} \dots \log 4.46102$	
<hr/> $V = 790.97 \text{ m.s.} \dots \log 2.89816$	

Using  $V$  to the nearest tenth of a meter-second, we now enter Table VI at the page headed  $25^\circ$  Range, and with the arguments  $V = 791.0$  and  $X = 28,908$  find the corresponding value of  $\text{Log } C$  as follows

$$\begin{array}{l} \text{For } V = 791 \text{ and } \text{Log } C = 1.135 \\ \hline X = 28.607 + .1 \times 548 = 28,662 \end{array}$$

$$\begin{array}{l} \text{For } V = 791 \text{ and } \text{Log } C = 1.155 \\ \hline X = 29.143 + .1 \times 565 = 29,199 \end{array}$$

$$\begin{array}{l} \text{For } V = 791 \text{ and } X = 28,908 \\ \hline \text{Log } C = 1.135 + \frac{246}{537} \times .020 = \underline{1.14416.} \end{array}$$

From this value of  $\text{Log } C$  we now deduce  $i$  by means of formula (406) transposed to the form

$$i = \frac{w}{\delta C d^2} \tag{805}$$

$w = 2099 \dots$	$\log 3.32201$
$\delta = .961 \dots$	$\log 9.98272 - 10 \dots \text{colog } 0.01728$
$C = \dots$	$\log 1.14416 \dots \text{colog } 8.85584 - 10$
$d^2 = 256 \dots$	$\log 2.40824 \dots \text{colog } 7.59176 - 10$
$i = \underline{.61220} \dots$	$\log 9.78689 - 10$

815. The problem of determining the range-table values for the angle of departure  $25^\circ$  is simply the reverse of the above, using the  $i$  just determined, and the standard range-table values of  $V$ ,  $w$ , and  $\delta$  in place of the observed values that applied to the experimental ranging. The procedure is as illustrated in the following example.

*Given:* The 16''2600 f.s. gun, for which the standard weight of projectile is 2100 lbs.; the coefficient of form, as determined by experimental ranging at the angle of departure  $25^\circ$ , is .61223.

*Find:* The range-table values of range (Col. 1), angle of fall (Col. 3), time of flight (Col. 4), striking velocity (Col. 5), and maximum ordinate (Col. 8), for the above gun at angle of departure  $25^\circ$  (Col. 2).

Example of computation of range-table values

$$C = \frac{w}{\delta i d^2} \tag{406}$$

$w = 2100$ .....	log 3.32222
$\delta = 1.00$ .....	colog 0.00000
$i = .61223$ .....	colog 0.21309
$d^2 = 256$ .....	colog 7.59176 - 10
$C =$ .....	log 1.12707
$V = 2600$ f.s.....	log 3.41497
(art. 705).....	log 9.48402 - 10
$V = 792.49$ m.s.....	log 2.89899

The arguments for Table VI and Table VII are therefore  $\phi = 25^\circ$ ,  $V = 792.5$  m.s., and  $\text{Log } C = 1.12707$ , and all that remains is to get the required values from these tables with the given arguments and to convert them to English units where necessary.

From Table VI we have

**From the table headed  $\phi = 25^\circ$  Range**

Range  
(Column 1)

For $V = 790$ ,	$X = 28,069 + .604 \times 538 = 28,394$
For $V = 800$ ,	$X = 28,601 + .604 \times 554 = 28,936$
For $V = 792.5$ ,	$X = 28,394 + \frac{1}{4} \times 542 = \underline{28,530 \text{ m.}}$

**From the table headed  $\phi = 25^\circ$  Time of Flight**

Time of flight  
(Column 4)

For $V = 790$ ,	$T = 5758 + .604 \times 38 = 5781$
For $V = 800$ ,	$T = 5819 + .604 \times 38 = 5842$
For $V = 792.5$ ,	$T = 5781 + \frac{1}{4} \times 61 = 5796$ or <u>57.96 sec.</u>

**From the table headed  $\phi = 25^\circ$  Angle of Fall**

Angle of fall  
(Column 3)

For $V = 790$ ,	$\omega = 4379 - .604 \times 44 = 4352$
For $V = 800$ ,	$\omega = 4396 - .604 \times 45 = 4369$
For $V = 792.5$ ,	$\omega = 4352 + \frac{1}{4} \times 17 = 4356$ or <u><math>36^\circ 18'</math></u>

**From the table headed  $\phi = 25^\circ$  Terminal Velocity**

Striking  
velocity  
(Column 5)

For $V = 790$ ,	$v_\omega = 4223 + .604 \times 73 = 4267$
For $V = 800$ ,	$v_\omega = 4244 + .604 \times 75 = 4289$
For $V = 792.5$ ,	$v_\omega = 4267 + \frac{1}{4} \times 22 = 4273$ or <u>427.3 m.s.</u>

From Table VII, using the pages headed  $V = 760$  m.s. and  $V = 800$  m.s. we have

Maximum  
ordinate  
(Column 8)

For $V = 760$ ,	$y_s = 3815 + .541 \times 111 = 3875$
For $V = 800$ ,	$y_s = 4153 + .541 \times 127 = 4222$
For $V = 792.5$ ,	$y_s = 3875 + .812 \times 347 = \underline{4157 \text{ m.}}$

Converting to English units, we have

$X = 28,530$ m.....	log 4.45530
$y_s = 4157$ m.....	log 3.61878
$v_\omega = 427.3$ m.s.....	log 2.63073
(art. 705).....	log 0.03886 · <u>log 0.51598</u> · <u>log 0.51598</u>
$X = 31,201$ yds.....	log 4.49416
$y_s = 13,638$ ft.....	log 4.13476
$v_\omega = 1401.9$ f.s.....	log 3.14671

These results may now be tabulated as follows, in units and number of places as in the U. S. Navy range tables.

Range	Angle of departure	Angle of fall	Time of flight	Striking velocity	Maximum ordinate
1	2	3	4	5	8
yards 31,201	°   '   '' 25   00.0	°   '   '' 36   18	seconds 57.96	f.s. 1402	feet 13,638

816. Very close agreement will be noted between the above results and those tabulated in the 16''2600 f.s. range table (see *Range and Ballistic Tables, 1935*), except in the case of the maximum ordinate, for which the latter gives 13,555 feet as compared to 13,638 feet as found above. This difference (83 feet) is due principally to the fact that the untranslated French edition of the A.L.V.F. Tables (1921) was used for calculating the maximum ordinate for this range table, whereas we have used the War Department Tables for this element (see arts. 702, 703, and 714). As will be noted from inspection of Columns 1 and 8 of the range table, a difference of 83 feet in the maximum ordinate corresponds to a difference of about 60 yards in the range, or about  $\frac{1}{5}\%$  of the range.\* The other very slight differences between the above results and the range-table values are due to smoothing operations that enter into the final tabulation of the latter, which will be described presently.

817. In the same manner as illustrated above for the angle of departure  $25^\circ$ , values of  $i$  are determined for each of the angles of departure at which experi-

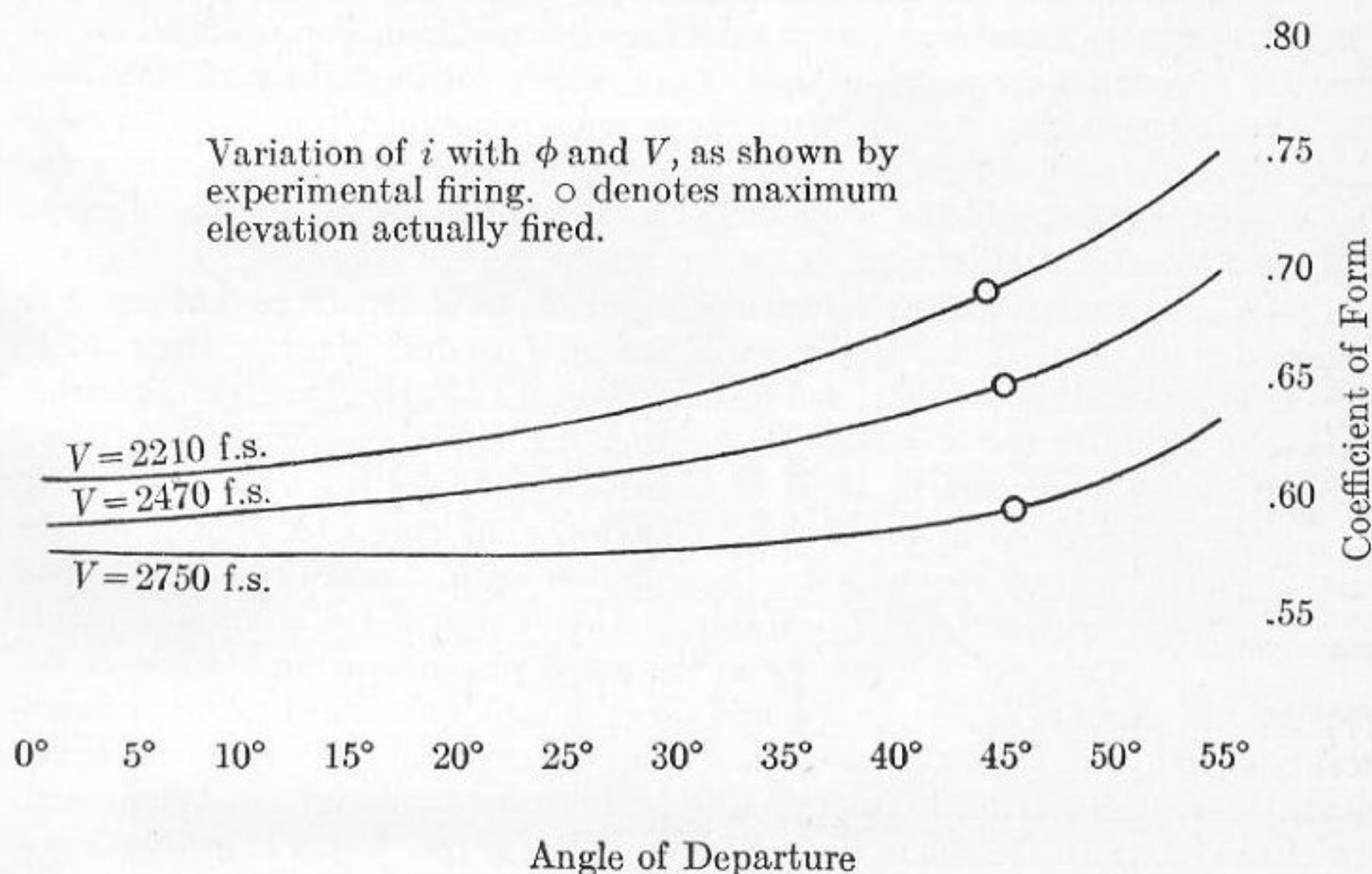


FIGURE 16

\* It should be understood, however, that this difference results from the use of the same value of  $i$  for two different tables. Exact agreement with the value  $y_0 = 13,555$  feet given by the A.L.V.F. Tables (1921) can be secured by using  $i = .6261$  in connection with Table VII in this case. On the whole, the agreement between results obtained from the two tables with the same value of  $i$  is remarkably good, considering the differences that exist between the retardation functions, altitude-density functions, values of gravity, and computational methods, on which the two systems of tables are based (ref. arts. 623, 702, 714).

mental ranging is done. A good estimate of the values of  $i$  pertaining to angles of departure not included in the experimental ranging may then be found by interpolating between the experimentally determined values; for this purpose a graph of  $i$  plotted against  $\phi$ , faired through the several points established by the experimental ranging, is most convenient.

For reasons outlined in article 417, it is evident that the value of  $i$  may be expected to vary with both  $V$  and  $\phi$ . Figure 16 illustrates the magnitude of such variations, as determined experimentally for a certain 16'' gun at full charge and at two reduced charges, the design and weight of projectile remaining the same. In this particular case the variation in  $i$  is inappreciable up to about  $\phi = 35^\circ$  for the full-charge initial velocity, while for the reduced initial velocities it becomes noticeable above about  $\phi = 10^\circ$ . This may be attributed to the fact that the combination of projectile and rifling design produce a more uniform flight stability for the maximum initial velocity than for the others. In any event, the experimental ranging reveals the conditions as they actually exist, and the necessary values of  $i$  for various angles of departure may be found from graphs, such as illustrated in Figure 16, drawn to large scale.

818. It is to be noted that the variation of  $i$  with initial velocity is great enough to require separate experimental rangings for changes of initial velocity that result from the use of different charges (such as the regular service charge and the reduced target-practice charges). It should be apparent, also, that separate experimental rangings and separate range tables may be required for different projectiles fired from the same gun, even though the initial velocities remain the same. In some cases the differences in projectiles do not affect their flight performance, and hence separate range tables are not required. For example, target-practice projectiles are designed to conform closely to the flight performance of the service projectiles, and the same range table is applicable to both. In other cases the difference between projectiles is very great, as in the case of common and flat-nose projectiles, and the range table for one type of projectile is then practically worthless for another type.

819. Having determined values of  $i$  for all angles of departure that are to be included in the range table, the process of finding the values of range, angle of fall, time of flight, striking velocity, and maximum ordinate for any angle of departure is exactly as illustrated in article 815.\* Range-table values are computed in this manner for 5°-increments in angle of departure (i.e., for the tabular values of  $\phi$  given in the A.L.V.F. Tables). The computed values for the several elements are then plotted against angle of departure, to large scale, and a graph is faired through the computed points for each element. From the graph of range against angle of departure, the angle of departure is found for each tabular value of range (in increments of 100 yards). The values picked from the graph are adjusted in order to make the second differences run smoothly; this serves to check the accuracy with which the intermediate points were picked from the graph. A similar process is followed for the other elements, the latter being tabulated, eventually, against range as argument. The general character of these graphs is illustrated in Plate IV.

The process of fairing curves through the computed points, and the subse-

\* See articles 513-514, and foot-note appearing under article 714. The work for angles of departure below  $15^\circ$  is essentially the same as here outlined except, of course, that the computed points are based on Siacci's Method. In view of the prospective adoption, in the near future, of numerical integration ballistic tables for all angles of departure, it is considered desirable to confine further references to the applications of Siacci's Method, at this time, to an Appendix.

quent process of smoothing out the second differences of values picked from the curves, naturally give rise to slight differences between results that eventually are tabulated in the range table and those which are found by computation for a specific point. This explains the slight differences which have been noted in article 816.

820. Although consideration of the drift of the projectile which results from the spin imparted to it by the rifling of the gun is not to be taken up at present, it is appropriate to mention at this point that the computation of the drift depends eventually upon experimental determination of a drift coefficient ( $D'$ ) which, in character and purpose, is not unlike the coefficient of form  $i$ .

Determination  
of observed  
drift

The drift coefficient must be given such a value that, when substituted in the drift formula (901), it causes agreement between the computed drift and the observed drift. The observed drift is

determined at the experimental ranging by correcting the observed lateral deviation (see art. 803 (d)) for the effect of wind across the line of fire. The problem of finding the drift coefficient from the observed drift will be taken up in Chapter 9.

### EXERCISES\*

1. *Given:* In the experimental ranging of a 16'' gun for the determination of range-table data for the service velocity of 2600 f.s., the following observed values were obtained from a group of five shots fired at angle of departure  $35^\circ$ : observed initial velocity, 2592 f.s.; uncorrected observed range, 36,026 yards; observed weight of projectiles, 2103 lbs.; observed ballistic density 1.035. The corrections to be applied to the observed range were found to be as follows: for ballistic wind, (+) 151 yards; for height of gun, (-) 7 yards; for curvature of the earth, (-) 89 yards.

*Find:* (a) The coefficient of form that corresponds to the above data. (b) The range-table range for this gun at angle of departure  $35^\circ$ .

*Answers:* (a)  $i = .61296$

(b)  $X = 36,801$  yards.

2. *Given:* In the experimental ranging of a 16'' gun for the determination of range-table data for the target-practice velocity of 2000 f.s., the following observed values were obtained from a group of five shots fired at angle of departure  $35^\circ$ : observed initial velocity, 1985 f.s.; uncorrected observed range, 23,779 yards; observed weight of projectiles, 2096 lbs.; observed ballistic density, .984. The corrections to be applied to the observed range were found to be as follows: for ballistic wind (+) 127 yards; for height of gun (-) 9 yards; for curvature of the earth, (-) 42 yards.

*Find:* (a) The coefficient of form that corresponds to the above data. (b) The range-table range for this gun at the angle of departure  $35^\circ$ .

*Answers:* (a)  $i = .67920$

(b)  $X = 23,994$  yards.

3. *Given:* The initial velocity, diameter, weight, and coefficient of form of the projectile, and the angle of departure.

*Find:* The values for the following range-table columns for the given angles of departure: Column 1 (Range); Column 3 (Angle of fall); Column 4 (Time of flight); Column 5 (Striking velocity); Column 8 (Maximum ordinate).

\* Use  $V$  to the nearest tenth of a meter-second in all cases.

	Given					Answers				
	$V$ f.s.	$d$ in.	$w$ lbs.	$i$	$\phi$	$X$ yds.	$\omega$	$T$ sec.	$v_{\omega}$ f.s.	$y_s^*$ ft.
A	2600	16	2100	.61230	15°	22,926	21° 34'	37.10	1485	5571
B	2600	16	2100	.61200	30	34,328	42 19	67.65	1423	18,589
C	2600	16	2100	.61140	40	38,630	51 48	85.76	1504	29,842

\* See art. 816.

4. From the 16''2600 f.s. range table given in *Range and Ballistics Tables, 1935*, deduce the values of  $i$  that correspond to the data of this range table at the following angles of departure: (a) 20°; (b) 35°.

Answers: (a)  $i = .61244$   
(b)  $i = .61280$

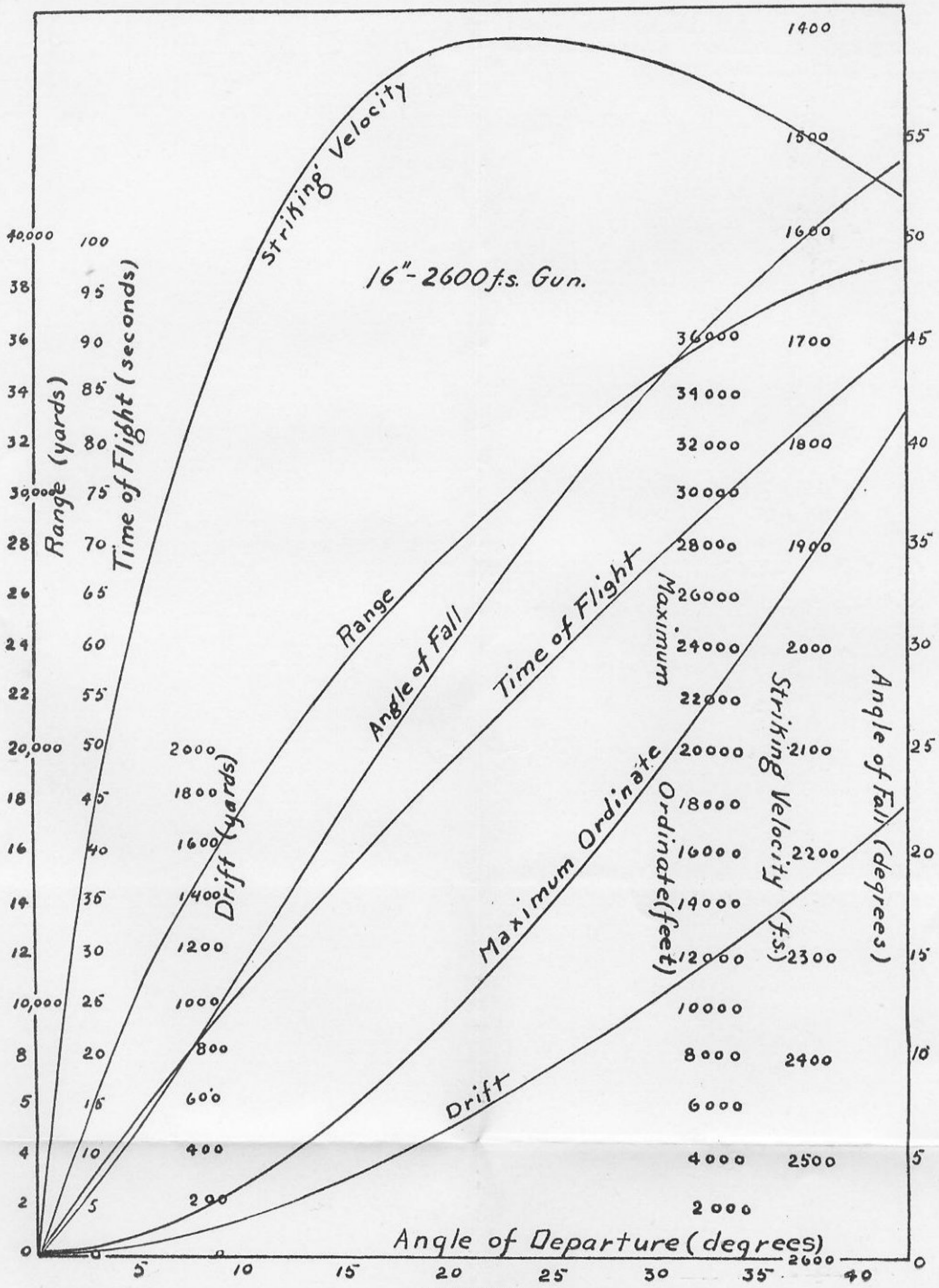


PLATE IV

THE DEVIATION OF THE TRAJECTORY FROM A PLANE CURVE;  
 THE DRIFT. THE DANGER SPACE. CHANGE IN HEIGHT OF  
 IMPACT. THE DETERMINATION OF THE QUANTITIES IN  
 COLUMNS 6, 7, AND 19 OF THE RANGE TABLE.

New Symbols Introduced

- $D$  . . . The drift of the projectile.  
 $D'$  . . . The drift coefficient used in determining the drift.  
 $\mu$  . . . The final twist of the rifling of the gun, i.e., the length of bore in calibers corresponding to one complete turn of the rifling at its final twist.  
 $S$  . . . The danger space (art. 914).

901. Having proceeded in the foregoing discussions under the assumption that the axis of the projectile remains coincident with the tangent to the trajectory (art. 213 (5)), we have developed solutions for the trajectory considered as a plane curve confined to the vertical plane containing the line of departure. We know, however, that the trajectory actually is not a plane curve, and that it deviates from the vertical plane through the line of departure by ever increasing

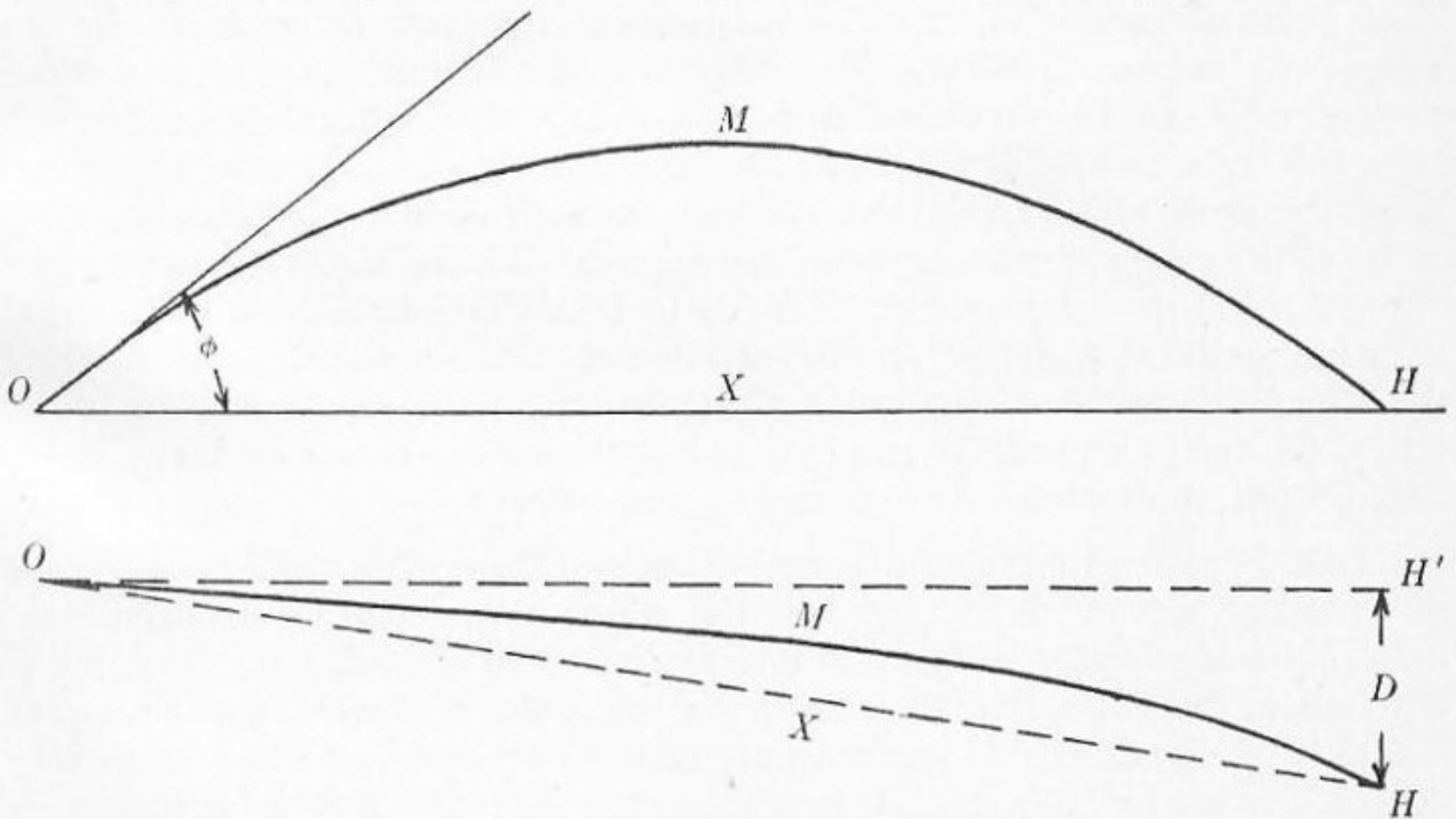


FIGURE 17

amounts as the point of fall is approached. In other words, the tangent to the trajectory constantly changes its direction not only with respect to the horizontal plane, but also with respect to the original plane of fire. This is illustrated in Figure 17, in which the upper diagram shows the side elevation and the lower diagram the plan view of the same trajectory. In the plan view,  $OH'$  represents the trace of the original plane of fire, and  $OMH$  the trace of the trajectory. The amount of the deviation of the curve  $OMH$  from the straight line  $OH'$  depends both on the lateral component of any wind that may be acting on the projectile, and on lateral

forces which result from its rotation. The amount of this deviation at the point of fall, under the standard condition of no wind, is called the *drift* and denoted by the symbol  $D$  (ordinarily expressed in yards). *The drift, then, is the lateral deviation of the point of fall from the original plane of fire, due only to effects set up by the rotation of the projectile.* Other lateral deviations, such as that due to wind, should not be termed as being part of the drift. The drift is measured as illustrated in Figure 17, where  $D$  is the perpendicular distance of the actual point of fall  $H$  from the original plane of fire  $OH'$ .

**902.** A study of Figure 17 will show that the point of fall  $H'$ , as predicted by a computation based on the assumption of a trajectory remaining within the original plane of fire, is located at a considerable lateral distance from the actual point of fall  $H$ . The lateral error  $D$ , however, although itself of considerable magnitude, may nevertheless cause only a slight error in the computed range. In other words, the difference between the computed range  $OH'$  and the actual range  $OH$  may be slight even if the distance  $H'H = D$  is considerable. The following examples bear this out. The 16'' 2600 f.s. range table shows that for this gun the drift is 38.3 yards at the range 10,000 yards; in this case the difference between  $OH'$  and  $OH$  (Fig. 17) is insignificant (actually only .08 yard). For trajectories limited to angles of departure not exceeding  $45^\circ$ , the drift does not exceed about 5% of the range, in which case the difference between  $OH'$  and  $OH$  is limited to about 0.1%.

It should be clear, then, that no appreciable error has been occasioned by our assumption of a plane trajectory up to this point, insofar as determination of the range is concerned; and it is to be presumed that a similar situation exists with respect to the other elements which have been dealt with under the same assumption. It is to be noted, however, that the difference between computed and actual results, as dealt with above, is in fact only a theoretical difference, since the value of  $i$  that is used for obtaining the computed range is based on an observed range referred to the actual point of fall (art. 803 (d)). Whatever distinction theoretically may exist between  $OH'$  and  $OH$  of Figure 17 therefore is fully accounted for by  $i$ . The same is true of any range effects which ensue from the rotation of the projectile, due to vertical forces set up by the projectile's obliquity to its direction of flight; this has already been discussed in article 417.

**903.** In proceeding to a study of the causes of the drift of a projectile, it may be remarked, first of all, that the complete explanation of this phenomenon rests on conjecture probably to a greater degree than is the case with any other phase of ballistics. One finds, in connection with some details of this explanation, a very considerable divergence of opinion among eminent authorities.\* The explanations offered here are but very general in character, and are based on a broad view of the situation. What may be a satisfactory explanation of the behavior of a rotating projectile in one specific instance, may fail to apply in another. The expository treatment given herein gives consideration to a number of hypotheses under which the causes of the drift have been investigated with the aid of mathe-

\* Exhaustive mathematical treatments of this subject are given on pp. 308-360, *Handbook of Ballistics*, Vol. I, Cranz and Becker; on pp. 172-257, *New Methods in Exterior Ballistics*, Moulton; in *The Aerodynamics of a Spinning Shell*, Fowler, Gallop, Lock, and Richmond (Philosophical Transactions of the Royal Society of London, Vol. 221, 1921); and in *The Damping Effect on a Rotating Projectile Due to the Path of the Center of Gravity*, Guion (Journal of the Maryland Academy of Sciences, Vol. I, 1930).

mathematical analysis. These analyses are much too lengthy for inclusion in this text, but may be found in the references noted on page 106.

The drift of an elongated, rotating projectile may be considered to result from three causes, viz:

- |                 |   |
|-----------------|---|
| Causes of drift | (a) Gyroscopic action.  |
|                 | (b) The action of air adhering to the projectile. ( <i>Magnus [left]</i> )                            |
|                 | (c) The cushioning action of air banking up on one side of the projectile. ( <i>Poisson [right]</i> ) |

It is reasonably certain that the latter two causes have only a very minor effect as compared to the first, and we shall therefore dismiss them with a brief discussion of their nature.

904. The effect of air adhering to the rotating projectile is similar to that which is employed to propel a rotor ship; it is often referred to as the *Magnus Effect*.<sup>\*</sup> In the case of a projectile, the initial tendency of the projectile to maintain the original direction of its axis, while the tangent to the trajectory moves downward, causes the air stream to strike the projectile's under-side. With right-handed spin, the air adhering to the right-hand side of the projectile, moving at the peripheral speed of the latter, then opposes the air stream created by the projectile's flight, and the result is an increase of pressure on the right-hand side. Similarly, on the left-hand side there is a rarefaction, and the projectile tends to move to the side of lesser pressure, i.e., to the left. If the axis of the projectile should change its direction and point below the tangent, the effect would be reversed. As the axis does in fact change its position with respect to the tangent, as we shall see presently, the effect of the adhering air may result in deviations of the projectile first to one side and then to the other, as well as up and down, but the initial movement is to the left, for right-handed spin. The Magnus Effect, in addition to its well-known application to rotor ships, serves also to explain the curves of a pitched baseball, and the "hooks" and "slices" of a golf-ball. However, considering the small obliquity of a projectile with respect to the direction of its flight, only a small component of the velocity of the adhering air is presented to the air stream; this fact, plus the probable alternations in direction of the resulting movements of the projectile, point to the conclusion that the Magnus Effect has but a small part in accounting for the observed drift.

The theory of the so-called cushioning effect<sup>†</sup> also depends upon obliquity of the projectile. If the under-side is presented to the air stream, the air banks up against this side, forming a sort of cushion against which the spinning projectile rolls by virtue of the greater friction of this relatively dense air cushion as compared to the condition on the opposite side. What has been said with regard to alternations of direction of the Magnus Effect also applies here, but in the case of the cushioning effect the initial movement is to the right for right-handed spin; hence these two effects oppose each other. It is probable, however, that the latter effect is of even less importance in accounting for the drift than is the Magnus Effect.

905. It is generally accepted that the principal cause of the drift lies in the gyroscopic properties of the rapidly spinning projectile. According to the laws of

\* From G. Magnus who, in about 1850-52, made important contributions to the theory of drift. Magnus not only investigated the effect which now bears his name, but also was among the first to investigate the gyroscopic theory.

† Sometimes referred to as the *Poisson Effect*, after S. D. Poisson who, in 1839, first offered it as a contributory cause of the drift.

General features of gyroscopic action

the gyroscope, the projectile seeks, first of all, to maintain its axis in the direction of the line of departure. The center of gravity of the projectile, however, follows the curved path of the trajectory, and the instantaneous direction of its motion, at any point, is that of the tangent to the trajectory at that point. The projectile's tendency to maintain the original direction of its axis therefore soon results in leaving this axis pointed slightly above the tangent to the trajectory, and the force of the air resistance opposed to the flight of the projectile is then applied against the latter's underside. This is illustrated in Figure 18 (a), in which  $GT$  is the tangent to the trajectory at the point which is occupied by the center of gravity  $G$  of the projectile. If the center of pressure of the surface exposed to the action of the air resistance were so located as to lie always in line with the center of gravity  $G$ , the condition illustrated would continue, with the axis maintaining its original direction and making an ever increasing angle with the tangent.

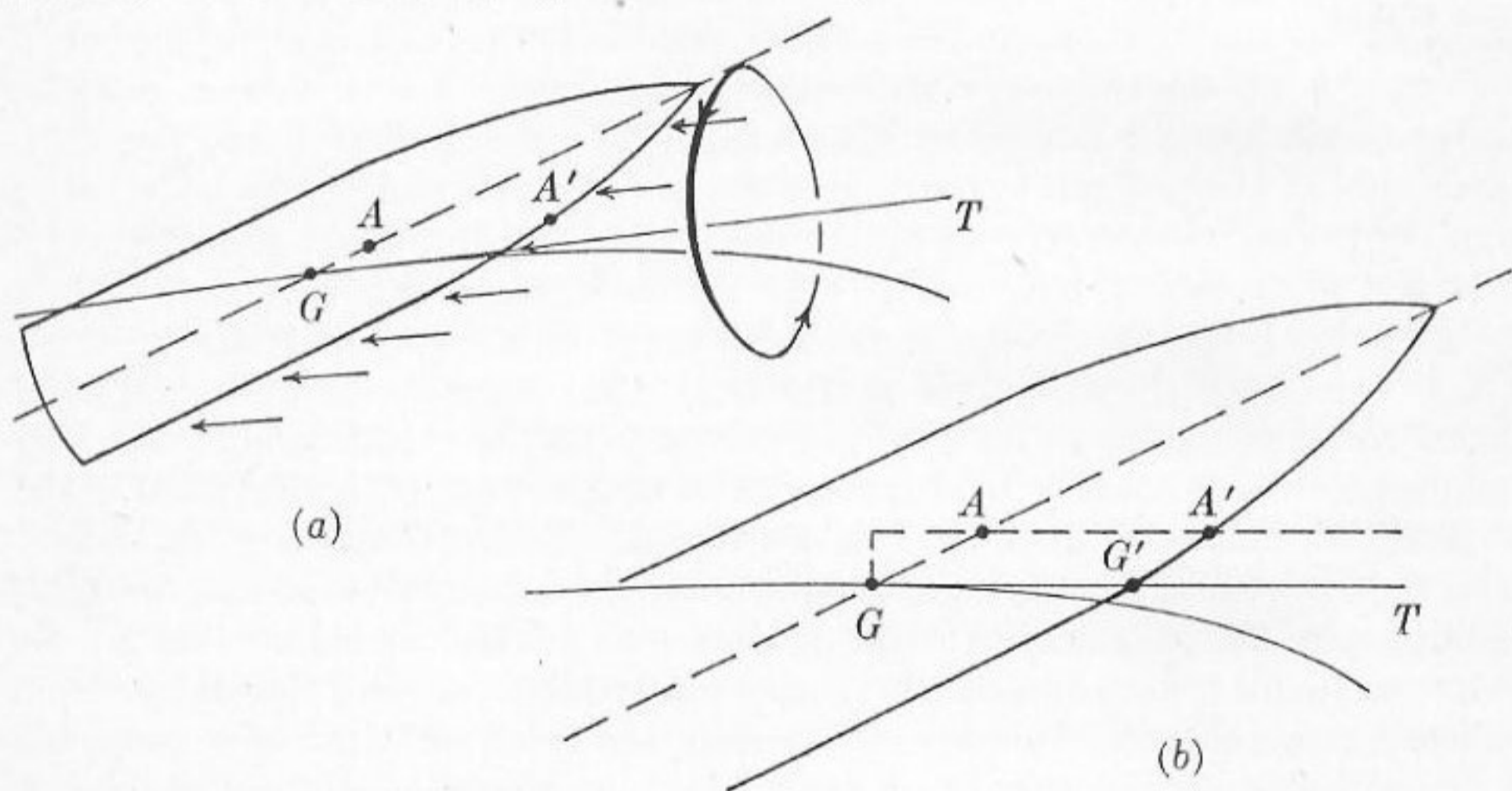


FIGURE 18

However, the relation between the weight distribution and surface of elongated, ogival-headed projectiles is such as to bring about the condition illustrated in Figure 18 (b). Here  $A'$  is the center of pressure of the surface that is being acted upon by the force of air resistance, and  $A'A$  is the line along which this force acts.  $A'A$  meets the axis of the projectile at  $A$ , which lies between the center of gravity and tip of the projectile. An overturning moment is thus set up, tending to tumble the projectile end over end, but the gyroscopic force of the projectile opposes this moment and tends to keep the axis in the original direction. According to the laws of the gyroscope, the action of the projectile in seeking to overcome this overturning moment must manifest itself in a precession of the projectile about the direction of the force which creates the moment; also according to these laws, the precessional revolution, for a projectile having right-handed spin, must be clockwise as viewed from the rear (see Figure 18 (a)).

The general features of gyroscopic action, as outlined above, can be demonstrated readily with any gyroscopic device, as for example a top. In 1911, the Bureau of Ordnance, U. S. Navy Department, made a small model projectile and suspended it in gimbals. The model was then rotated at a speed comparable to the speed of rotation of a projectile. To simulate the air resistance in flight, a blast of air was directed against the rotating model, and the latter's behavior confirmed the above

Laboratory reproductions of gyroscopic action

theory.\* As soon as the blast of air met the rotating model, the latter began to precess about the direction of the air. As the direction of the blast was changed to represent the changing direction of the air resistance in flight, the model's axis depressed to meet the new direction of the air blast.

906. Under the condition illustrated in Figure 18(a), i.e., with the projectile's axis already at a considerable angle with respect to the tangent, and the latter changing its direction downward at a slow rate, the projectile would undoubtedly make complete precessional revolutions about the tangent, as illustrated. However, the actual situation is not as simple as this. Assuming that the projection from the gun is perfectly regular, so that the projectile commences its flight exactly head-on, no precession at all takes place until the tangent drops below the projectile's axis. As soon as this occurs, the projectile's axis starts to move in the right-hand semicircle of its precessional revolution, and the axis therefore soon is moving downward, i.e., in the same direction as the tangent. The rate at which the axis moves, i.e., its precessional period, depends chiefly upon the projectile's physical features and rate of spin (it varies inversely as the latter). It is readily conceivable, therefore, that the character of the projectile's precessional motion with respect to the tangent depends materially upon the relation that exists between the rates at which the axis and tangent are changing direction.

The character of the motion of the axis with respect to the tangent

A general conception of this situation may be gained by studying the motion of the axis through a complete cycle. The tangent having dropped sufficiently below the axis to set up precession, the axis moves first to the right and then down. In the early part of the first quadrant of this revolution, the downward component of motion of the axis is less than that of the tangent, but as the axis turns farther its downward component increases, and eventually the axis overtakes and passes the tangent. As the end of the second quadrant is approached, the downward component of the axis eventually decreases again to the point where it becomes less than that of the tangent, and the latter then tends to overtake the axis.† Assuming, however, that the tangent does not fully overtake the axis at the end of the second quadrant, the latter then enters the third quadrant, moving to the left of the tangent and then up, and eventually completes the whole revolution. But in the entire left-hand semicircle of this revolution, the motion of the axis is contrary to that of the tangent. Hence this semicircle is of shorter duration than the right-hand semicircle, and it follows that the axis of the projectile must, in any event, be a longer time to the right of the tangent than to the left.

907. The very general analysis given above serves to prove that even if the projectile actually makes complete revolutions about the tangent, and deviates alternately to the right and left as the result of the pressure exerted against its sides, there is nevertheless a preponderance of such deviations toward the side on which the axis and tangent are both moving downward, and for right-handed spin this is to the right (with respect to the direction of flight). In short, it is the constant dipping of the tangent, in combination with the precessional motion of the axis, that accounts

Why the drift preponderates to one side

\* Ref. p. 36, *Ordnance Pamphlet No. 399* (U. S. Navy, 1912).

† Dr. Cranz, in his *Handbook of Ballistics*, Vol. I, (pp. 328-329), concludes that, "Neglecting nutation, the point of the shell describes in space a cycloidal curve, and the axis of the shell a cycloidal cone, lying on the right-hand side of the vertical plane through the tangent"; also, "And the fact that the point of the shell, after completing a cycloidal arc, always comes into coincidence with the tangent to the path, is the reason why the shell strikes the ground with its point." On pp. 345-350, he offers a graphical step-by-step solution of the motion of the axis with respect to the tangent, by means of which he supports these conclusions.

both for the drift and for the fact that the axis of a properly designed projectile never deviates very far from the tangent.

The fact that the axis and tangent are approaching coincidence in the second quadrant of the precessional revolution lends support to the theory that the motion of the axis, and hence the sidewise deviation of the projectile, are cycloidal in nature (see footnote, p. 109). According to this theory, the axis actually comes into coincidence with the tangent at the end of each half revolution, and there is no deviation at all to the left. In the absence of rigorous proof, it seems debatable whether this condition can be accepted as universally applicable, although it is readily conceivable that it may be approached to varying degrees, according to the immediate relation between the stability of the projectile and the curvature of the trajectory.

A conclusion arrived at by some eminent authorities, as the result of careful mathematical analysis, is that, as a consequence of the dipping of the tangent, the precessional motion tends to damp itself out and to approach a condition in which the projectile's axis, somewhere in the right-hand semi-circle of its precessional revolution, practically keeps pace with the downward motion of the tangent.\* Under this condition the axis then remains pointed slightly to the right of the tangent, and the projectile yaws continuously to the right. The angle at which the axis tends to steady itself, i.e., the angle of yaw, evidently depends upon the rate at which the tangent is moving downward, and hence must increase as the curvature of the trajectory increases. This is in accord with the observed fact that the drift increases not at a constant rate with respect to the range, but at an ever increasing rate. (It increases, in fact, approximately in proportion to the square of the time of flight.) The observed nature of the drift, in this respect, is accounted for also under the conception of cycloidal oscillations, or of complete precessional revolutions, of the projectile's axis about the tangent. For in the former case the duration of each yaw to the right, and in the latter the preponderance of the yaws to the right over those to the left, likewise depend on the rate at which the tangent is moving downward.

A close analysis of the behavior of a rapidly spinning projectile in flight is further complicated by the frictional effects already mentioned in article 904, and by effects resulting from irregular projection. The latter may have a very marked influence on the motion of the projectile, especially in the early stages of its flight. The projectile may emerge from the bore with its axis already slightly canted with respect to the tangent, or it may, while still in the region of the muzzle blast, be thrown aslant by the column of gases acting against its base. Precession results also from any such initial inclination of the projectile's axis, but obviously it does not progress in the same relation with respect to the motion of the tangent as does the precession which results directly from the latter. Also, the precessional motion that is set up by irregularities in the initial stages of flight eventually is damped out by the regular precession, which alone is sustained by a continuing cause.

**908.** The gyroscopic stability of a projectile can be controlled within fairly

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\* In *New Methods in Exterior Ballistics* (p. 254), Dr. Moulton concludes that, "The oscillations of the projectile are damped toward the solution given in . . . , and in this solution, for guns having right-hand rifling, the axis of the projectile is to the right of the plane of fire and a little below the tangent to the trajectory." A similar result is stated in *The Aerodynamics of a Spinning Shell*, by Fowler, Gallop, Lock, and Richmond.

General considerations governing the degree of stability desired in a projectile

wide limits by means of the initial spin given to the projectile by the gun, i.e., by means of the twist of the rifling. Other things remaining the same, a greater rate of spin causes greater stability. It is not to be supposed, however, that great stability (in the sense here implied) is a desirable feature, for with infinitely great stability the projectile would resist precession altogether

and consequently would fail to fly head-on. With relatively very great stability, precession would be very slow, the amplitude of the precessional arc relatively great, and the lateral deviation excessive and perhaps erratic; moreover, under this condition the irregularities of flight incident to canted ejection would persist for a longer time and consequently produce greater errors in the point of fall. What is to be desired, rather, is a degree of stability sufficiently great to resist the overturning moment (Figure 18(b)), but not so great as to prevent precession of a small amplitude about the tangent. The attainment of a satisfactory balance in these respects rests chiefly on proving-ground experiments designed to afford immediate information as to the flight characteristics of projectiles. These experiments involve the firing of projectiles through a series of cardboard screens, and careful measurement of the holes left by them in the several screens, whence the obliquity of the projectiles at these several points in their flight may be deduced.\* Regularity in the points of fall at all ranges is, of course, the ultimate criterion as to the acceptability of a given design.

909. The attainment of an ideal degree of stability for a projectile is complicated by the fact that the conditions which define this ideal do not remain constant throughout the trajectory. As the remaining velocity of the projectile decreases, the overturning moment (Figure 18 (b)) and hence the degree of stability required to oppose this moment, also decrease. If the relative stability of the projectile is to remain uniform throughout the trajectory, it is necessary, therefore, that the rate of spin decrease in the same proportion as the remaining velocity decreases. However, this is not the case, since the rate of spin decreases less rapidly than does the remaining velocity, and consequently the relative stability of the projectile increases as the latter proceeds in its flight. Moreover, as the projectile ascends to air of lesser density the overturning moment decreases, and the relative stability increases in the ascending branch of the trajectory due also to this cause. This is of special significance in the case of trajectories having a large angle of departure, since in this case the increased stability of the projectile in the vicinity of the maximum ordinate is particularly undesirable in view of the relatively great rate of change of curvature in that region. The design of a projectile, with respect to its stability characteristics, therefore must be a compromise between the requirements for short trajectories and long trajectories. †

Apart from the lateral effects which result from the oscillations of the projectile's axis about the tangent to the trajectory and which manifest themselves in the drift, vertical effects evidently also result therefrom. That is, the varying angle of inclination of the axis with respect to the tangent causes deviations in the

\* A description of such experiments is given in *Ordnance Pamphlet No. 399* (U. S. Navy, 1912), and in *The Aerodynamics of a Spinning Shell*, Fowler, Gallop, Lock, and Richmond.

† Experiments have recently been made with projectiles whose ogives are fitted with wind vanes, with a view to creating greater uniformity in the stability of the projectile throughout the trajectory by causing its rate of spin to be retarded more nearly in proportion to the retardation of the velocity of translation.

vertical plane as well as in the horizontal plane, and the rotation of the projectile therefore affects the range. The angle of inclination between the axis and tangent also modifies the effective cross-sectional area of the projectile, which affects the retardation, and hence the range. Variations in the mean effective stability factor for various trajectories of the same gun and projectile, arising from the considerations outlined in the foregoing paragraph, thus contribute largely to the variations in coefficient of form with angle of departure that are noted from the results of experimental ranging. In short, the rotation of the projectile, although it is commonly associated only with the phenomenon of drift, has a very important bearing on the range as well.

**910.** Analytical expressions for the drift have been deduced under the hypothesis of gyroscopic action, but so little is known about the actual values of some of the physical factors involved in them, under the varying conditions that exist in a whole trajectory, that the practical solution for the drift eventually depends on expressions that are highly empirical in nature. As has already been observed above, the behavior of the projectile under the influences set up by its rotation accounts largely for the empirical nature of the coefficient of form,  $i$ , which enters into the equation to the trajectory as referred to the vertical plane. Considering, now, that this behavior of the projectile is but one of several factors that influence its path with respect to the vertical plane, whereas it is the sole cause of the lateral deviation from this plane (under the assumed condition of no wind),

it can readily be appreciated that the entire system of solution for this lateral deviation is empirical to a much greater degree than is the case with the solutions for elements that have previously been dealt with. The relations indicated by an analytical treatment of the drift under the gyroscopic theory, are incorporated to varying degrees in the numerous expressions for this element that have been proposed from time to time. But it is characteristic of all such expressions that the analytical terms contained in them are usually rather broad approximations, based on averages pertaining to entire trajectories and sometimes even on averages pertaining generally to all projectiles of similar type. And, in practically all cases, the expressions depend also on purely empirical coefficients which must be determined from the results of experimental firing quite in the same manner as is the case with the coefficient of form,  $i$ .

**911.** The drift formula that has been used for the more recent of the range tables that appear in *Range and Ballistic Tables, 1935*, is given here as an example.\* This formula is due to Colonel A. Hamilton, U. S. Army.

\* A drift formula devised by Mr. E. B. Scott, of the U. S. Naval Proving Ground staff, has been used in connection with some of the later range tables. The general expression for this formula is

$$D = \frac{T^2}{a + \frac{b\mu}{V} + \frac{\phi}{c}}$$

in which  $a$ ,  $b$ , and  $c$  are coefficients that are to be determined by experiment. The values  $a = .871$ ,  $b = 349.4$ , and  $c = 128.2$  have been found to give very satisfactory agreement with observed results, under a wide variety of conditions, with projectiles  $3\frac{1}{2}$  calibers long (these values require  $\phi$  to be expressed in degrees and give  $D$  in yards). For other projectiles different values of  $a$ ,  $b$ ,  $c$ , are required, but satisfactory results are obtained also by applying an additional coefficient (of the nature of  $D'$ ) to the entire expression, with values of  $a$ ,  $b$ ,  $c$  as given above. For additional formulas, see pp. 350-356, *Handbook of Ballistics*, Cranz and Becker; also pp. 118-121, *Computation of Firing Tables for the U. S. Army*, H. P. Hitchcock.

The very simple formula  $D = kT^2$ , in which  $k$  is a constant that is to be determined

Hamilton's  
drift  
formula

$$D = X(1 - D') \frac{d^3}{\mu w} (\phi^\circ + \omega^\circ) \sec \phi. \quad (901)$$

In the above formula  $\phi$ ,  $\omega$ ,  $d$ ,  $w$ , and  $X$  have their usual meanings, but the values of  $\phi$  and  $\omega$  within the bracket are to be expressed in radians.  $D$  is the drift, in yards or in feet according to whether  $X$  is expressed in yards or feet. The factor  $\mu$  represents the final twist of the rifling of the gun, expressed in terms of the length of bore *in calibers* covered by one complete turn of the rifling at its final twist.

The factor  $D'$ , which we shall call the *drift coefficient*, is purely empirical in nature. Its value is so chosen that when it is substituted in the drift formula (901),

it will cause agreement between the computed drift and the observed drift as determined by experimental ranging. It should be apparent, then, that the principles governing the determination of  $D'$  are quite similar to those which are involved in the determination of  $i$ . Also, that the value of  $D'$  thus determined partakes of the character of  $i$ , i.e., it accounts for any influences which have contributed to the observed drift but have not separately been accounted for in the drift formula. The effect of lateral jump, for example, is included in the value of  $D'$ . The value of  $D'$ , however, depends chiefly on the form of the projectile, and it may be thought of as a sort of coefficient of form for drift. Its value for U. S. Navy projectiles is usually around .7.

912. In the course of experimental ranging, the uncorrected observed lateral deviation is determined as already explained in article 803 (d); by applying to this a correction for the component of wind across the line of fire that existed at the time of the ranging, the observed drift is found. The latter is then substituted

in (901) and the required value of  $D'$  is found. However, since we have no means for correcting this observed drift as would be necessary to refer it to the *corrected* observed range, we use in (901) the range at which the observed drift actually was measured, i.e., the *uncorrected* observed range, and for  $\phi$  and  $\omega$  we use the values corresponding to the latter as determined from subsequent computations of the range-table data. An example will illustrate these features.

*Given:* In the experimental ranging of the 16'' 2600 f.s. gun the following average results were obtained for a group of five shots fired at angle of departure  $25^\circ$ : uncorrected observed range, 31,845 yards; observed weight of projectiles 2099 lbs.; observed lateral deviation, 747 yards to the right; correction for observed ballistic wind component of 4 knots across the line of fire, 42 yards to the right. By subsequent computation the angle of departure for the range 31,845 yards was found to be  $25^\circ 57'$ , and the angle of fall  $37^\circ 27'$ . The final twist of rifling in the gun was one turn in 32 calibers.

*Find:* The drift coefficient  $D'$ .

First we shall transpose (901), and at the same time introduce the relation  $57.3 = 1$  radian in order that the values of  $\phi$  and  $\omega$  in the bracket may be expressed in degrees. We have then

$$(1 - D') = \frac{D}{X} \times \frac{\mu w}{d^3} \left( \frac{57.3}{\phi + \omega} \right) \cos \phi \quad (902)$$

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experimentally and  $D$  and  $T$  have their usual meanings, has also been found to give very satisfactory results in connection with long projectiles, and has been used in computing Column 6 of some tables.



horizontal dimension, in no way depends upon the trajectory; it remains the same for all guns and at all ranges. The *height* of the target, however, contributes to this total range allowance an amount which depends on the angle of inclination of the trajectory near the point of fall, and which varies, therefore, according to both the gun and the range, and must be calculated accordingly.

The total range allowance with which we are dealing here may be thought of either as defining the limits within which the sight-bar range of the gun may vary, considering the target to remain at a fixed distance; or as defining the limits within which the distance to the target may vary, considering the sight-bar range to remain fixed. The amount of the allowance differs somewhat depending upon which of the above conceptions is applied in determining it, but the distinction is of no importance except at very short ranges, and need be considered only in connection with certain special problems which are met at Short Range Practice and which will be dealt with separately in Appendix B.

The total range allowance corresponding to the dimensions of the target, when considered in the sense of an allowable variation in target distance and of a fixed sight-bar range, is called the *danger space* and denoted by  $S$  (usually expressed in yards).<sup>\*</sup> According to this conception we may then state that *the danger space for a given target and trajectory is the greatest distance through which the target may be moved in the line of fire and still be intersected at some point by that trajectory*. This is illustrated in Figure 19, in which  $OMH$  is a trajectory whose point of fall is at the fixed range  $OH$ . A target, whose height is  $h$  and whose horizontal dimension in the line of fire is  $l$ , is shown in the position in which the trajectory strikes just at the waterline of

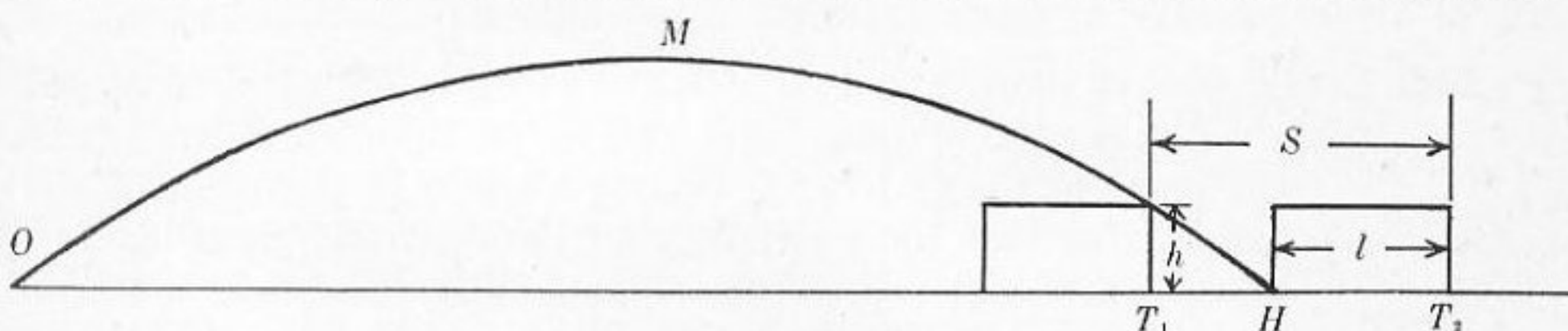


FIGURE 19

the side nearer the gun, and the position in which the same trajectory just fails to clear the upper edge of the side farther from the gun. The danger space,  $S = T_1T_2$ , is the distance between these two positions. From the construction of the figure, it is evident that the total danger space is composed of two parts,  $HT_2 = l$  being the horizontal dimension in the line of fire, and  $T_1H$  being the projection of the height  $h$  upon the horizontal plane.

915. The value of that portion of the danger space which depends only on the *height* of the target (as  $T_1H$  in Figure 19), is tabulated in Column 7 of the range table; the value tabulated applies to a target height of 20 feet, and the value for any other heights (within reasonable limits) may be found therefrom by simple proportion (see also art. 920). In other words, *Column 7 of the range table gives the value of the danger space for a target whose height is 20 feet and whose horizontal dimension in the line of fire is zero*. It is to be understood, of course, that the usual range-table conditions apply here as elsewhere, and that the base of the target is there-

<sup>\*</sup> When taken in the opposite sense, i.e., as an allowable variation in sight-bar range at a fixed target distance, this quantity is termed the *hitting space*. The terms *danger space* and *hitting space* ordinarily are practically interchangeable, except at very short ranges.

fore considered to be in the horizontal plane which passes through the gun (art. 801).

In Figure 20,  $AB$  represents a target whose height is  $h$  and whose horizontal dimension in the line of fire is zero, and  $S = AH$  is the danger space for the target against the trajectory  $OBH$ , whose horizontal range is  $X = OH$ , and maximum ordinate is  $y_s$ . The danger space is thus seen to be the difference between the greatest distance,  $X = OH$ , and shortest distances,  $x = OA$ , at which the given

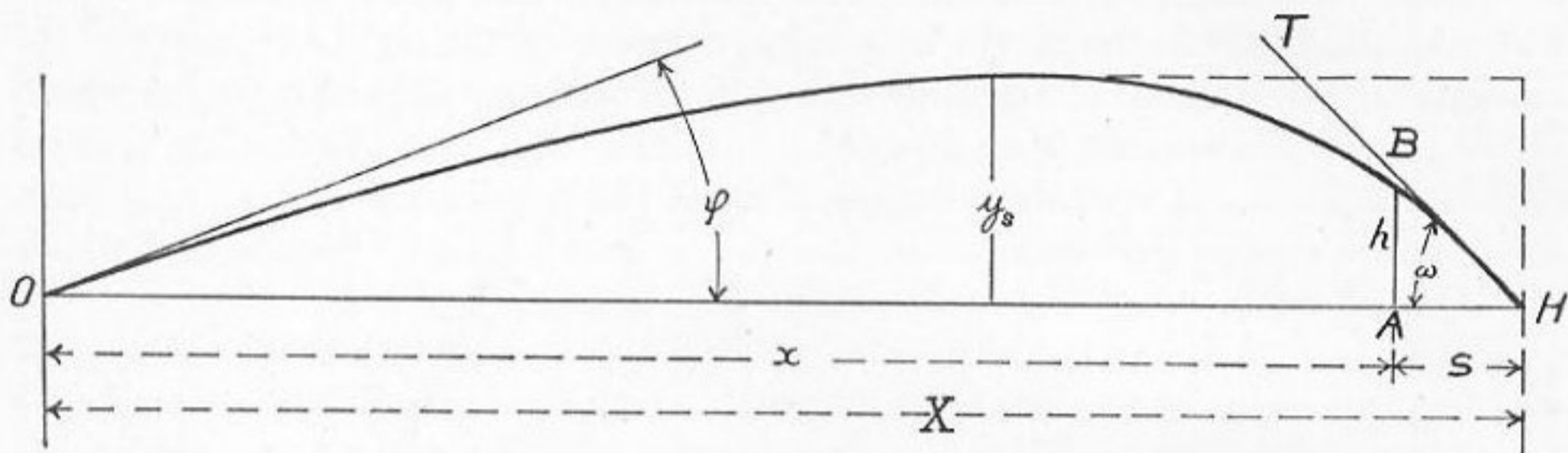


FIGURE 20

target may be situated and yet be intersected by the given trajectory. The direct solution for the danger space therefore involves the determination of the abscissa  $x$  of the trajectory corresponding to the ordinate  $y = h$ , whence  $S = X - x$ . A solution according to this principle may be made, of course, by finding the point  $x, y = h$ , by numerical integration. Satisfactory approximations of the danger space may be made, however, by methods which are much less laborious than the above

916. Except at very short ranges the value of  $h$  (Figure 20), within practical limits of target height, is small in comparison with the maximum ordinate  $y_s$ , and the point in the trajectory where  $y = h$  is near the point of fall. Considering that the trajectory, for the relatively short distance between the point  $y = h$  and the point of fall, is practically a straight line which coincides with the tangent to the trajectory at the latter point, we have the approximate relation

Formula for Column 7 for moderate and long ranges

$$S = h \cot \omega \quad \text{mod + long ranges} \quad (903)$$

in which  $\omega$  is the angle of fall. The above formula gives a satisfactory approximation of the danger space (Col. 7) for ranges at which the angle of fall exceeds about  $3^\circ$  (about 4000-6000 yards for most naval guns). The application of (903) is illustrated in the following example.

Given: The 16" 2600 f.s. gun, range 10,000 yards, angle of fall  $5^\circ 55'$ .

Find: The danger space (range-table Column 7).

Since the target height for Column 7 is 20 feet, we shall use  $h = 20$  feet. Then we have

$h = 20$ feet . . . . .	$\log 1.30103$
$\omega = 5^\circ 55'$ . . . . .	$\text{lcot } 0.98450$
$S = 192.99$ feet . . . . .	$\log 2.28553$
<u><math>= 64</math> yards</u>	

917. At shorter ranges the point  $y = h$  (Fig. 20) may be so far from the point of fall that material error may result from considering the portion of the trajectory

Process for  
computing  
Column 7  
for short  
ranges

beyond this point to be a straight line. In such cases a solution may be made by a process that is somewhat indirect, but that is simple and sufficiently accurate for any practical purpose. In Figure 21 the trajectory  $OBH$ , whose horizontal range is  $OH$ , is shown just touching the top of the target  $AB$ , whose height is  $h$  and whose horizontal dimension in the line of fire is zero; then  $S = AH$  is the danger space for the target height  $h$  at the range  $OH$ . There is shown also the trajectory whose point of fall is just at the bottom of the target  $AB$ , and whose horizontal range therefore is  $OA$ . The difference between the horizontal ranges,  $OH$  and  $OA$ , respectively, of these two trajectories, evidently is equal to the danger space for the longer of

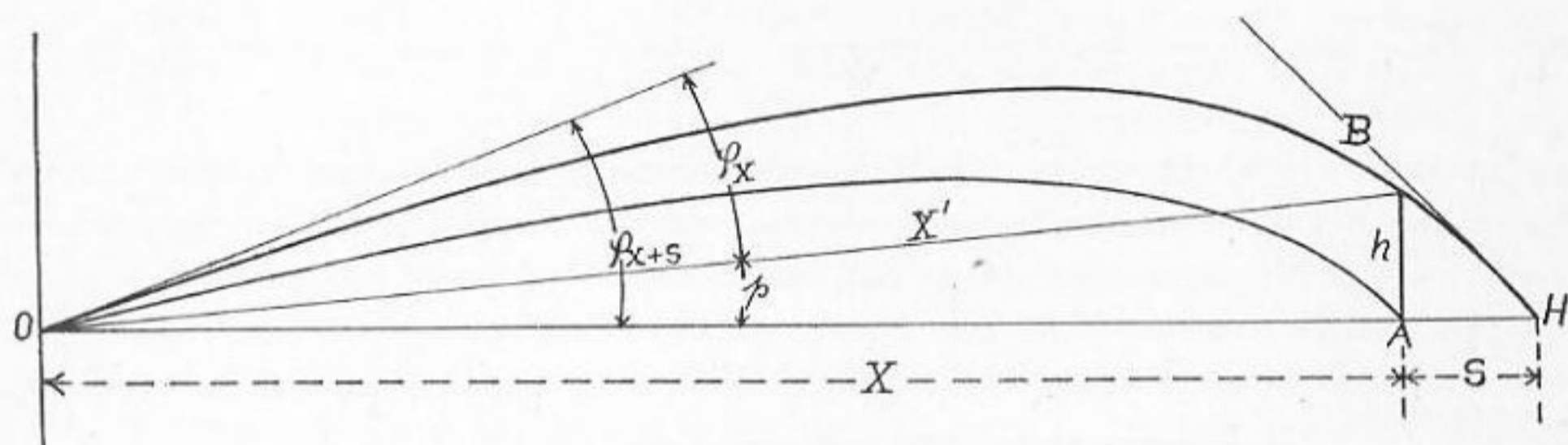


FIGURE 21

them. Let us assume, now, a value  $OA = X$  for the horizontal range of the trajectory whose point of fall is at the bottom of the target, and find how much must be added to this to give the trajectory which just touches the top of the same target; this amount evidently equals the danger space  $S$  of the latter trajectory. The assumed range of the shorter trajectory being  $X$ , we then have the danger space  $S$  for the longer trajectory, i.e., for the range  $X + S$ .

Since  $h$  is always small in comparison with  $X$ , we may consider that the inclined range  $X' = OB$  is equal to the horizontal range  $X = OA$  (for example, at the shortest range tabulated, 1000 yards, and for a 40-foot target, which is as high as we have occasion to consider, the difference between  $OB$  and  $OA$  is about 0.1 yard). Also, the angle of position  $BOA = p$  is always small in any practical case (for example,  $p = 46'$  in the rather extreme case just cited). Under these conditions the theory of rigidity of the trajectory is applicable (art. 319), and we may consider that the trajectory  $OB$  represents the trajectory  $OA$  tilted upward by the amount of the angle  $p$ . Since the two trajectories are considered to be similar in all respects, the angle of elevation of the trajectory  $OB$  is equal to the angle of departure of the trajectory  $OA$ , i.e., to the angle of departure corresponding to the horizontal range  $X$ . We shall therefore denote the angle of elevation of trajectory  $OB$  by  $\phi_X$ , as illustrated. Likewise, since the horizontal range of the trajectory  $OH$  equals  $X + S$ , we shall denote its angle of departure by  $\phi_{X+S}$ , i.e., the angle of departure corresponding to the horizontal range  $X + S$ .

From the figure it is apparent that the angle  $p$  is defined by the relation

$$\tan p = \frac{h}{X} \quad (904)$$

whence, with a given range  $X$  and target height  $h$ ,  $p$  may be found. From Columns 1 and 2 of the range table, the value of  $\phi$  corresponding to any value of  $X$  may be found, and hence the value of  $\phi_X$  is available directly. As shown in the figure,

$$\phi_{X+S} = \phi_X + p \quad (905)$$

and with  $\phi_{X+S}$  we can find in the range table the corresponding range, which, in this case, is  $X+S$ . The required danger space is then

$$S = (X + S) - X. \quad (906)$$

*It is important to note, however, that the value of  $S$  so found applies not to the assumed range  $X$ , but to the range  $X+S$ .* It is for this reason that the process is indirect. A direct solution according to the same principle is not possible without resort to successive approximations. For if we seek to determine the value of  $S$  corresponding directly to an assumed value of  $X$ , we cannot find  $p$ , since in this case we

must take  $\tan p = \frac{h}{X-S}$ . The process as actually used also requires successive

approximations to be made, insofar as the solution for the danger space corresponding to a particular range is concerned. In the computations for range-table values, however, this is of no consequence, since we make solutions for many ranges, plot the values of danger space against the ranges to which they apply, and then take from the graph the values of the danger space corresponding to the tabular values of range.

918. For further demonstration of the process outlined above, let us apply it to the case of the 16" 2600 f.s. gun, and assume  $X = 1350$  yards = 4050 feet.

From the 16" 2600 f.s. range table we find that at 1350 yards the angle of departure is 34' 0; hence we have  $\phi_X = 34' 0$ . Since Column 7 of the range table is based on a target height of 20 feet, we have  $h = 20$  feet, whence, from (904),

Examples of  
computation of  
Column 7 for  
short ranges

$$\tan p = \frac{20}{4050}, \quad \text{and} \quad p = 17' 0.$$

Also, from (905),

$$\phi_{X+S} = 34' 0 + 17' 0 = 51' 0$$

and from the range table we find that this angle of departure corresponds to the range 2004 yards, or  $X+S = 2004$  yards. Then, from (906),

$$S = 2004 - 1350 = \underline{654 \text{ yards.}}$$

The result thus obtained is, that for a 20-foot target the danger space is 654 yards at the range 2004 yards; in other words, the point of fall of the given trajectory being at a distance of 2004 yards from the gun, a 20-foot target can be moved from this point of fall toward the gun a distance of 654 yards, or to a point only 1350 yards from the gun, before its top falls below the trajectory.

Now let us assume  $X = 2000$  yards = 6000 feet, and  $h = 20$  feet. Then

$$\tan p = \frac{20}{6000}, \quad \text{and} \quad p = 11' 5$$

$$\phi_{X+S} = 50' 9 + 11' 5 = 1^{\circ}02' 4$$

$$X + S = 2430 \text{ yards}$$

$$S = 2430 - 2000 = \underline{430 \text{ yards}}$$

whence we find that at the range 2430 yards the danger space for the 20-foot target is 430 yards.\*

919. It will be observed in Figure 20 that if the height of the target is equal to or greater than the maximum ordinate, then the target may be moved the entire distance from the point of fall to the gun, i.e., the danger space is equal to the whole range. The maximum range at which this occurs, for a given height of target, is called the *danger range*, and it is evidently the range at which the maximum ordinate just equals the given height of target. It will be noted that in the range table the values in Columns 1 and 7 are the same whenever the value in Column 8 (maximum ordinate) is 20 feet or less. It will be clear that if the danger range, or any range less than the latter, is set on the sights of a gun, and the gun is sighted at the bottom of the target, then the target may be located at any distance from the gun equal to or less than the range set on the gun, and yet be hit. This condition holds true, however, only if the gun and target are in the same horizontal plane, as is assumed for the range table. Actually, the values in Column 7 which are equal to the whole range have no practical value, since the gun is in fact always above the water. For the same reason, other values from Column 7 are not applicable directly to the usual situation at short ranges, in which the gun's height above the water is comparable to the height of the maximum ordinate itself.

920. At moderate and long ranges, the values of the danger space (Col. 7) for target heights other than 20 feet, within the limits of the latter that are likely to occur in practice, may be found by applying simple proportion to the values tabulated in Column 7. At short ranges, however, very great errors may result from such a process. For example, in the 16"2600 f.s. range table, at the range 1300 yards, the maximum ordinate is 8 feet, and the danger space for a 20-foot target (Col. 7) is therefore 1300 yards. This value evidently applies also to any target height equal to or greater than 8 feet; but for a target height the least bit less than 8 feet the danger space at once decreases to a value less than one-half as great as the above. Again, at 2600 yards the danger space for the 20-foot target is 368 yards, but for a 40-foot target it is 2600 yards, since the maximum ordinate at this range is just 40 feet. It is evident, then, that proportionality between danger spaces for the same trajectory, with respect to their corresponding target heights, vanishes altogether when either of the target heights involved equals or exceeds the maximum ordinate of the given trajectory. But even for target heights that remain within the limits of the maxi-

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\* The process just outlined is the one now in use for computing Column 7. For range tables computed before about 1926, a very much less accurate process was employed in connection with Column 7. The formula previously used for short ranges was,

$$S = h \cot \omega \left( 1 + \frac{h \cot \omega}{X} \right).$$

Since this formula is now obsolete, its derivation will not be given here, but it may be found in earlier editions of this book. It is worthy of note, however, that in many of the range tables still in current use at this writing, Column 7 represents values found by the above formula. This is also the case with all of the tables given in *Range and Ballistic Tables, 1935*, which accounts for the discrepancies between the values found above and those given in the 16" 2600 f.s. range table. This table gives  $S = 527$  yards at range 2004 yards, and  $S = 402$  yards at range 2430 yards, whereas the results found above are, respectively, 654 yards and 430 yards. It will be observed that the discrepancy is very considerable at the shorter range but much less serious at the longer range. Above about 3000 yards the difference becomes immaterial; this applies to all guns. The 16"2600 f.s. range table in current use in the fleet contains the values as found by the more accurate process given above.

mum ordinate, the use of simple proportion in connection with Column 7 is subject to errors which increase as either of the target heights involved approaches the height of the maximum ordinate; this is very likely to be the case at short ranges, where ordinary target heights are comparable to the heights of the maximum ordinates. For the reasons outlined in the present and foregoing paragraphs it must be concluded, therefore, that the values in Column 7 are practically useless at short ranges. Methods for obtaining values that are useful in connection with certain special short-range problems will be taken up in Appendix B.

**CHANGE IN HEIGHT OF IMPACT (COLUMN 19)**

921. Column 19 gives the change in height of impact that results from a variation of 100 yards in the sight-bar range. Such information is useful, for example, in direct-flight spotting, when the sight-bar range is adjusted by observing the point of impact in the vertical plane of the target. Let us suppose that the situation is as illustrated in Figure 21, and that the point of impact is observed to be at a height  $h$  feet above the waterline of the target  $AB$ , the sight-bar range set on the gun being equal to  $OH$ . If this point of impact is to be lowered to the waterline, the sight-bar range evidently must be reduced to  $OA$ . The problem is similar to that of finding the danger space for a given target height. In the latter case we found the change in range corresponding to a given height  $h$ ; in the present case we are to find the height  $h$  that corresponds to a given change of range, 100 yards being the amount of the change assumed for Column 19.

We shall denote the required change in height of impact by  $\Delta h$ , and the change of range to which it corresponds by  $\Delta X$ . Since  $\Delta X = 100$  yards, which is small in comparison with the whole range  $X$  in any practical case, we may assume without material error that the portion of the trajectory from the point where  $y = \Delta h$  to the point of fall is a straight line, and that

Formula for  
Column 19 of  
the range table

$$\Delta h = \pm \Delta X \tan \omega \tag{907}$$

$\omega$  being the angle of fall at the assumed range  $X$ . The sign  $\pm$  has been introduced in order to indicate that the formula is applicable both for increases and for decreases in  $X$ , i.e., for  $\Delta X = \pm 100$  yards. From Figure 21 it will appear that for the case of the increase in  $X$ , the value of  $\omega$  should be found with the range  $X + \Delta X$ , rather than with  $X$  directly. However, the change in  $\tan \omega$  corresponding to a change of 100 yards in range is sufficiently small, in any case, to have an inappreciable effect on the value of  $\Delta h$  that is to be found, and we may therefore use (907) with the value of  $\omega$  corresponding to the assumed  $X$  whether we are dealing with an increase or a decrease in the latter. The application of (907) is illustrated in the following example.

*Given:* The 16"2600 f.s. gun, range 10,000 yards, angle of fall  $5^\circ 55'$ .

*Find:* The change in height of impact corresponding to a variation of 100 yards in sight-bar range.

$\Delta X = \pm 300$ feet . . . . .	$\pm \log 2.47712$
$\omega = 5^\circ 55'$ . . . . .	$\text{ltan } 9.01550 - 10$
$\Delta h = \pm 31$ feet . . . . .	$\pm \log 1.49262$

The result obtained means that, for the 16"2600 f.s. gun at the range 10,000 yards, an increase or decrease of 100 yards in the sight-bar range of the gun will cause the point of impact in the vertical plane of the target to be raised or lowered,

respectively, 31 feet. It will be observed that the same information is available from Column 7, for which the computation has been made in article 916. Since, from the latter, a target height of 20 feet corresponds to a

Relation between Columns 7 and 19

change of 64 yards in range, we may deduce that the height  $\frac{100}{64} \times 20$

= 31 feet corresponds to a change of 100 yards. This relation between Columns 7 and 19 exists at all but very short ranges.

EXERCISES

1. *Given:* The diameter and weight of the projectile, the range, angle of departure, angle of fall, drift coefficient  $D'$  and final twist of rifling  $\mu$ .

*Find:* The values for Column 6 (drift) of the range table for the given ranges.

		Given							Answers.
		$d$	$w$	$X$ (yards)	$\phi$	$\omega$	$D'$	$\mu$	$D$ (yards)
A	5''-3150 f.s.	5	50	5000	2° 01'	2° 52'	.7869	25	9.1
B		5	50	8000	4 14	7 31	.7869	25	35.1
C		5	50	10000	6 26	12 48	.7869	25	72.0
D		5	50	15000	14 59	30 52	.7869	25	264.8
E	16''-2600 f.s.	16	2100	3000	1 18	1 22	.67	32	2.8
F		16	2100	5000	2 14	2 27	.67	32	8.2
G		16	2100	8000	3 49	4 23	.67	32	23.1
H		16	2100	12000	6 11	7 40	.67	32	58.7
I		16	2100	20000	12 15	17 10	.67	32	211.3

2. *Given:* The range and the angle of fall.

*Find:* The values for Column 7 of the range table for the given ranges (danger space for a 20-foot target). (Use formula 903.)

		Given		Answers
		$X$ (yards)	$\omega$	$S$ (yards)
A	5''-3150 f.s.	5000	2° 52'	133
B		8000	7 31	51
C		10000	12 48	29
D		15000	30 52	11
E		22000	57 33	4
F	16''-2600 f.s.	5000	2 27	156
G		12000	7 40	50
H		20000	17 10	22
I		39000	52 53	5

3. *Given:* The range and the angle of fall.

*Find:* The values for Column 19 of the range table for the given ranges (change in height of impact for a variation of 100 yards in the sight-bar range).

		Given		Answers
		X (yards)	$\omega$	$\Delta h$ (Col 19) (feet)
A	5"-3150 f.s.	5000	2° 52'	15
B		8000	7 31	40
C		10000	12 48	68
D		15000	30 52	179
E		22000	57 33	472
F	16"-2600 f.s.	3000	1 22	7
G		8000	4 23	23
H		12000	7 40	40
I		15000	10 44	57
J		20000	17 10	93

4. Proceeding as illustrated in article 918, make computations for  $S$  with the ranges 1600 yards and 1800 yards (16"-2600 f.s. gun) and for a target height of 20 feet. Using the two values of danger space (Col. 7) and their corresponding ranges as thus determined, and using also the two sets of values already found in article 918, plot a graph of danger space against range (scale 1" = 100 yards), and from this graph determine the values of the danger space for the ranges 2000, 2100, 2200, 2300, and 2400 yards.

*Answers* (see footnote on p. 119):

<u>Range</u> (yds.)	<u>Column 7</u> (yds.)	<u>Range</u> (yds.)	<u>Column 7</u> (yds.)
2004	654	2000	657
2148	548	2100	580
2281	481	2200	518
2430	430	2300	471
		2400	440

THE DETERMINATION OF THE EFFECTS OF VARIATIONS FROM RANGE-TABLE STANDARDS FOR INITIAL VELOCITY, WEIGHT OF PROJECTILE, AND ATMOSPHERIC DENSITY (RANGE-TABLE COLUMNS 10, 11, and 12).

New Symbols Introduced

- $\Delta X_V$  . . . . The change in range due to a variation in the initial velocity.  
 $\Delta X_C$  . . . . The change in range due to a variation in the ballistic coefficient (or in any factor contained in the latter, such as  $\delta$  and  $w$ .)  
 $M$  . . . . A multiplier to be used with Column 12 of the range table (art. 1014).  
 $\Delta V_w$  . . . . The change in initial velocity due a variation in the weight of the projectile.  
 $\Delta C_w$  . . . . The change in the ballistic coefficient due to a variation in the weight of the projectile.  
 $\Delta X_w$  . . . . The change in range due to a variation in the weight of the projectile.  
 $m$  . . . . A coefficient used in the formula for finding  $\Delta V_w$  (1003).

1001. Having examined the problem of determining the values of elements of the trajectory under the standard conditions assumed for range tables (art. 801), our next problem is to consider the means by which these values may be adjusted for variations from the assumed standards. Although such variations affect all elements of the trajectory, our chief concern is to determine their effects on the location of the point of fall. The changes in range due to variations in initial velocity, weight of projectile, and atmospheric density, and of wind, motion of gun, and motion of target in the line of fire; and the lateral deviations due to wind, motion of gun, and motion of target across the line of fire; are given in Columns 10-18 of the range table. In the present chapter we shall deal with Columns 10, 11, and 12, which give the changes in range due to variations, respectively, in initial velocity, weight of projectile, and atmospheric density.

The effect of a variation from standard in any of the factors on which the trajectory depends, evidently can be found by making a complete solution of the trajectory for the non-standard combination, and comparing it with the solution corresponding to the standard combination. For example, if the change ( $\Delta X_V$ ) in range, corresponding to a reduction ( $\Delta V$ ) from the standard initial velocity, is to be found, then  $\Delta X_V$  evidently is the difference between the ranges corresponding to the combinations  $\phi, V, C$ , and  $\phi, (V - \Delta V), C$ , the values of  $\phi$  and  $C$  being the same for both cases. Similarly, the variation in range corresponding to a variation from standard in any of the factors contained in  $C$  can be found by comparing solutions made with the standard and non-standard values of  $C$ , the values of  $\phi$  and  $V$  remaining the same. This direct method is particularly advantageous for determining variations in the terminal elements, since the values of the latter, for any desired combination of  $\phi, V$ , and

General principles involved in determining effects of variations from standard conditions

$C$ , can be found readily in ballistic tables. We shall therefore use this method in finding the values for Columns 10, 11, and 12.\*

### CHANGE IN RANGE DUE TO A GIVEN VARIATION IN INITIAL VELOCITY (COLUMN 10).

1002. Practice has varied as to the size of velocity variation assumed for the tabular values in Column 10. In some of the older tables that are still in use, the value in Column 10 corresponds to a 50 f.s. variation in initial velocity, but the later practice is to tabulate in this column values that correspond to a 10 f.s. variation. The heading of Column 10 indicates in all cases the size of variation to which the tabulated values apply. Practice has varied also as to the sign of the velocity variation assumed in deriving the values for this column. Formerly it was the practice to make these values equally applicable either to increases or to decreases of velocity. This was done by taking the mean of values found, respectively, with plus and minus variations in  $V$ . The more recent practice is to base Column 10 only on a minus variation in  $V$ , since variations from the standard  $V$  are practically always of this sign. Erosion, of course, always causes reductions in  $V$ ; variations in  $V$  due to non-standard powder temperature also are reductions in practically all cases, since the standard powder temperature, 90°F., is also the highest temperature permitted in magazines.

The heading of Column 10 in the more recent tables states that the values in this column are the changes in range corresponding to a (+) 10 f.s. variation in initial velocity. This is not intended to imply that the values were *derived* on the basis of a plus variation only, but merely to indicate that the sign of the tabulated values corresponds to an increase of initial velocity (the plus sign is to be understood when no sign appears before the tabulated values). No confusion should arise as to the proper sign to be used with these values, since an increase in  $V$  always causes an increase in range, and a decrease in  $V$  always a decrease in range. Information as to the practice followed in deriving the values in this column may be found in the introductory pages of the range table. All of the extracts included in *Range and Ballistic Tables, 1935*, are from the older range tables, in which Column 10 is based on  $\pm$  variations in  $V$ .

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\* By partial differentiation of the fundamental equations of the trajectory, with respect to each of the several factors ( $\phi$ ,  $V$ ,  $C$ ) on which the latter depends, it is possible to set up expressions which define the differential relations between any of these factors and any element of the trajectory. By differentiating the fundamental equations with respect to  $C$  alone, the differential relation between  $C$  and any element is found, and hence it is possible, for example, to express the relation between a variation in  $C$  and the effect of the latter on the values of  $x$ ,  $y$ ,  $t$ ,  $v$ ,  $\theta$ , at any point in the trajectory. By integrating these differential expressions, it is then possible to evaluate these effects within any desired limits. Thus the effect of a variation in  $\delta$  (and hence in  $C$ ) which is assumed to apply only within specified limits of altitude, can be found by integrating the appropriate differential expression between these limits. Practical applications of variations of this character will be dealt with presently, and reference will be made, in this connection, to the principles involved in the determination of such variations by the method just outlined, i.e., by the method of differential variations. A complete demonstration of this method is, however, much too lengthy to be included in this text. Comprehensive treatments of the differential-variation method are available in the following sources: *A Course in Exterior Ballistics* (War Department Document No. 1051, December, 1920), by R. S. Hoar; *The Method of Numerical Integration in Exterior Ballistics* (War Department Document No. 984, October, 1919), by D. Jackson; *New Methods in Exterior Ballistics*, by F. R. Moulton.

**1003.** The following example illustrates the later practice that is followed in the computation of values for Column 10. This example is based on the same gun and angle of departure assumed for the example given in article 815.

**Computation of Column 10**

*Given:* The 16"2600 f.s. gun,  $\phi = 25^\circ$ ,  $\text{Log } C = 1.12707$ .

*Find:* The change in range due to a variation of  $(-)$  10 f.s. in the initial velocity.

We shall find the ranges,  $X_1$  and  $X_2$ , corresponding to the velocities, respectively,  $V_1 = 2600$  f.s. and  $V_2 = 2590$  f.s., using the A.L.V.F. Tables, with  $\phi = 25^\circ$  and  $\text{Log } C = 1.12707$  in each case. The difference between these ranges, which we shall denote by  $\Delta X_V$ , then represents the change in range due to the given variation in  $V$ . Converting the velocities to metric units, and then entering the A.L.V.F. Tables, we have

$V_1 = 2600 \text{ f.s.} \dots \dots \log 3.41497$ (art. 705) $\dots \dots \log 9.48402 - 10$ $V_1 = 792.49 \text{ m.s.} \dots \dots \log 2.89899$	$V_2 = 2590 \text{ f.s.} \dots \dots \log 3.41330$ (art. 705) $\dots \dots \log 9.48402 - 10$ $V_2 = 789.44 \text{ m.s.} \dots \dots \log 2.89732$
--	--

From the table headed  $\phi = 25^\circ$  Range

For $V = 790$ ,	$X = 28,069 + .604 \times 538 = 28,394$
For $V = 800$ ,	$X = 28,601 + .604 \times 554 = 28,936$
For $V_1 = 792.5$ ,	$X_1 = 28,394 + .25 \times 542 = 28,530 \text{ m.}$
For $V = 780$ ,	$X = 27,541 + .604 \times 522 = 27,856$
For $V = 790$ ,	$X = 28,069 + .604 \times 538 = 28,394$
For $V_2 = 789.4$ ,	$X_2 = 27,856 + .94 \times 538 = 28,362 \text{ m.}$

$\Delta X_V = (-) 168 \text{ m.} \dots \dots \dots \log 2.22531$	
(art. 705) $\dots \dots \dots \log 0.03886$	

$\Delta X_V = (-) 184 \text{ yards.} \dots \dots \dots \log 2.26417$	
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This result differs only slightly from that given in Column 10 of the 16"2600 f.s. range table (*Range and Ballistic Tables, 1935*), the latter having been found by assuming a  $(\pm)$  10 f.s. variation.

**1004.** The differences with respect to  $V$ , as tabulated in the A.L.V.F. Tables for range, evidently are the values of  $\Delta X_V$ , in meters, corresponding to successive variations of 10 m.s. in  $V$ . By examining these differences we may readily determine what degree of error is occasioned by using simple proportion in connection with Column 10 to find the changes in range corresponding to velocity reductions considerably in excess of 10 f.s. For example, let us assume  $\phi = 40^\circ$ ,  $\text{Log } C = 1.135$ , and a standard  $V$  of 790 m.s. In this case we find, from the A.L.V.F. Tables, that the change of range corresponding to a reduction of 10 m.s. (i.e., from 790 m.s. to 780 m.s.) is 711 m., while the change of range corresponding to a reduction of 30 m.s. (i.e., from 790 m.s. to 760 m.s.) is 2109 m. If simple proportion is applied to the change for 10 m.s. in order to find the change for 30 m.s., we then find for the latter the value  $3 \times 711 = 2133$  m., which is in error by 24 m., or by a little more than 1% with respect to the correct value of  $\Delta X_V$ ; in comparison with the range (about 35,000 m.) the difference of 24 m. amounts to an error of less than one-tenth of one percent.\*

**Use of Column 10 for other than tabular variation in velocity**

\* For some range tables the value of  $\Delta X_V$  for  $(-)$  10 f.s., as tabulated in Column 10, has been derived by taking one-tenth of the value of  $\Delta X_V$  for  $(-)$  100 f.s. In such cases the situation discussed above is reversed, i.e., changes of range determined by applying simple proportion to the values given in Column 10 are somewhat more accurate for large velocity changes than for small ones.

The case just considered may be regarded as a rather extreme test as to the degree of error that may be incurred by applying simple proportion to Column 10 values; for smaller values of  $\phi$  and  $C$ , the degree of error becomes relatively even less. The comparison between 10 m.s. and 30 m.s. values, as made above, serves to indicate very nearly what the situation is with respect to 10 f.s. and 100 f.s. values. The fact that the ratio in the latter case is increased to 10 to 1, as compared with only 3 to 1 in the former case, is of no consequence, since the 10 f.s. value is in fact obtained by linear interpolation with respect to a 10 m.s. tabular interval. In other words, no greater inaccuracy is incurred by deriving the 100 f.s. value from the 10 f.s. value, than by deriving the 30 m.s. value from the 10 m.s. value.

Velocity reductions due to the combined effects of erosion and non-standard powder temperature ordinarily do not exceed 100 f.s., and rarely exceed 125 f.s. It may be concluded, therefore, that the process of applying simple proportion to Column 10 values for finding changes in range corresponding to any reductions in initial velocity that are likely to be encountered in practice, does not of itself entail any appreciable loss of accuracy.

1005. In all of the above determinations of the effect of velocity variations, it has been assumed, however, that the value of  $\Delta X_V$  depends only on the variation in velocity itself. That is to say, the values of  $\Delta X_V$ , for all velocity reductions considered, have been based on the same value of  $C$ . The situation is altered somewhat by the consideration that  $i$  (and hence  $C$ ) actually varies with the velocity. Such variations in  $i$ , if not accounted for, cause the values of  $\Delta X_V$  to be in error by amounts which increase with the size of velocity variations to which the values of  $\Delta X_V$  apply; this will remain true whether  $\Delta X_V$  is derived from Column 10 or directly from the ballistic tables. Such errors can be eliminated only by determining  $\Delta X_V$  experimentally, and this procedure has been followed in some cases.

1006. For some target practices, the initial velocity of the gun is very much less than the service initial velocity; for example, the target-practice velocity of the 16"/45 gun is 2000 f.s., or 600 f.s. less than the service velocity of 2600 f.s., and in the case of the 5"/51 gun the difference is 850 f.s. (3150-2300). These reductions in initial velocity are much too great to be handled by means of Column 10, and additional range tables are therefore prepared for the target-practice velocities; in some cases other special velocities also are provided for by means of additional range tables. A separate experimental ranging is conducted for each of these additional range tables, and changes in the value of  $i$  are thus fully accounted for. All other operations involved in the preparation of a range table are also performed separately for each of the tables.

#### CHANGE IN RANGE DUE TO A GIVEN VARIATION IN ATMOSPHERIC DENSITY (COLUMN 12).

1007. In all U. S. Navy range tables, Column 12 is based on a variation of ( $\pm$ ) 10% in atmospheric density. The heading of Column 12 in the more recent tables states that the values in this column are the changes in range corresponding to a (-) 10% variation in atmospheric density. This is not intended to imply that the values were derived on the basis of a minus variation only, but merely to indicate that the sign of the tabulated values corresponds to a decrease in density (the plus sign is to be understood when no sign appears before the tabulated values). No con-

Effect of changes in  $i$  on values derived from Column 10

Reduced-velocity range tables

Size and sign of variations assumed for Column 12

fusion should arise as to the proper sign to be used with these values, since an increase in atmospheric density always causes a decrease in range, and a decrease in atmospheric density always causes an increase in range.

1008. It is shown by formulas (404) and (406), respectively, that a given  $\pm\%$  variation in atmospheric density causes an equal  $\pm\%$  variation in the density factor  $\delta$ , and hence an equal  $\mp\%$  variation in  $C$ . The change in range due to a  $(\pm)$  10% variation in atmospheric density therefore can be found by determining the change in range due to a  $(\mp)$  10% variation in  $C$ . The problem is handled very simply, in accordance with the same general principle that has already been applied in connection with variations in  $V$ , i.e., by interpolation in the A.L.V.F. Tables. The process is illustrated in the following example, which is based on the same gun and angle of departure assumed for the example given in article 815.

*Given:* The 16"2600 f.s. gun,  $\phi = 25^\circ$ ,  $\text{Log } C = 1.12707$ .

*Find:* The change in range due to a variation of  $(\pm)$  10% in atmospheric density.

We shall find the ranges corresponding to values of  $C$  decreased and increased by 10% with respect to the standard value of  $C$ , using the A.L.V.F. Tables, with  $\phi = 25^\circ$  and  $V = 792.5$  m.s. (2600 f.s.) in each case. The difference between these ranges represents the change of range corresponding to a change of 20% in  $C$ , i.e., the sum of the changes of range due to a 10% decrease in  $C$  and to a 10% increase in  $C$ . One half of this sum then is the *mean* of the changes in range due to a 10% decrease and a 10% increase in  $C$ , and it may be termed the change in range due to a  $(\mp)$  10% variation in  $C$ , which corresponds to a  $(\pm)$  10% variation in atmospheric density. We shall let the symbol  $\Delta X_C$  denote a change in range due to a change in  $C$ , and in this case specifically a change in range due to a change in atmospheric density.

Proceeding as just outlined, we find the reduced and increased values of  $C$  as follows.

$C$ .....	log 1.12707.....	log 1.12707
.90.....	log 9.95424 - 10	
1.10.....	.....	log 0.04139
$C_1$ .....	log 1.08131	
$C_2$ .....	.....	log 1.16846

From the table headed  $\phi = 25^\circ$  Range

For Log $C = 1.075$ ,	$X = 26,988 + \frac{1}{4} \times 499 = 27,113$
For Log $C = 1.095$ ,	$X = 27,529 + \frac{1}{4} \times 516 = 27,658$
For Log $C_1 = 1.08131$ ,	$X_1 = 27,113 + .316 \times 545 = 27,285$ m.
For Log $C = 1.155$ ,	$X = 29,143 + \frac{1}{4} \times 565 = 29,284$
For Log $C = 1.175$ ,	$X = 29,679 + \frac{1}{4} \times 581 = 29,824$
For Log $C_2 = 1.16846$ ,	$X_2 = 29,284 + .673 \times 540 = 29,647$ m.

$$\Delta X_C = (\mp) \frac{1}{2} (29,647 - 27,285) = 1181 \text{ m.}$$

$$\Delta X_C = (\mp) 1181 \text{ m.} \dots \dots \dots \text{log } 3.07225$$

$$\text{(art. 705)} \dots \dots \dots \text{log } 0.03886$$

$$\Delta X_C = (\mp) 1292 \text{ yards}^* \dots \dots \dots \text{log } 3.11111$$

\* This agrees with the value given in the latest 16"2600 f.s. range table, which is not available for publication in *Range and Ballistic Tables, 1935*. The range tables given in the latter volume are obsolete, which accounts for many of the differences noted between results given in these tables and results found according to the computations given in this text.

1009. As has been noted in the procedure by which the above value of  $\Delta X_c$  has been found, the latter represents the mean of the changes in range due to a 10% decrease and a 10% increase in  $C$ . The range corresponding to the standard  $C$  of the above problem has already been found to be 28,530 m. (art. 815); the ranges corresponding to a 10% decrease and a 10% increase in  $C$  are, respectively, 27,285 m. and 29,647 m. (art. 1008). The change in range for the 10% decrease in  $C$  therefore is  $28,530 - 27,285 = 1245$  m., and for the 10% increase it is  $29,647 - 28,530 = 1117$  m. If the mean of these (1181 m.) is used for the value of Column 12, the latter is then 64 m. or about 5% too small for the 10% decrease in  $C$ , and 64 m. or about 6% too great for the 10% increase in  $C$ ; in comparison with the whole range, however, the error in either case is only about 0.2%. The situation is substantially the same at greater and lesser ranges, i.e., the values in Column 12 are somewhat too small for decreases in  $C$  and somewhat too great for increases in  $C$ . It follows that, with respect to variations in atmospheric density, the values in Column 12 are somewhat too small for increases in density, and somewhat too great for decreases in density.

In actual practice, especially at sea, variations from the standard atmospheric density rarely amount to as much as 10%, and the degree of inaccuracy involved in the use of Column 12, as noted above, therefore represents rather wide extremes. Ordinarily variations in atmospheric density are confined within comparatively narrow limits, and the errors incident to the use of Column 12 are limited accordingly. In any event, it is probable that the degree of inaccuracy involved in the use of Column 12 is always relatively small in comparison with the degree of inaccuracy involved in the practical determination of the atmospheric density itself, since the latter, at best, is always subject to rather broad approximations (art. 425).

1010. From what already has been said with regard to the character of variations in atmospheric density (art. 420), it is apparent that methods must be devised for discriminating between variations that pertain to the entire trajectory and those that apply only to portions of the trajectory. In other words, it is necessary to devise means for taking into account a given variation in atmospheric density that pertains only within specified limits of altitude, or, more generally, variations of different magnitude that apply within different limits of altitude.

Method of handling density variations that are not uniform throughout the trajectory

This problem can be handled by expressing the differential relation between  $X$  and  $C$ , and by integrating this expression within the required limits (see foot-note adjacent to article 1001). It is possible, in this manner, to find the effect on the trajectory as a whole of a variation in  $C$ , and hence in atmospheric

density, that applies only within specified limits of the ordinates of that trajectory. That is to say, a value of  $\Delta X_c$  can be found that expresses the change in range of the whole trajectory due to the effect of a ( $\pm$ ) 10% variation in atmospheric density on the portion of a trajectory which is included within any given zone of altitude, as for example for zones extending from the surface to a height of 600 feet, from the height 600 feet to the height 1500 feet, from 1500 feet to 3000 feet, etc.

The practical method of dealing with this problem is to compare the effects of a given density variation operating only on a limited portion of the trajectory, with the effect of an equal density variation operating on the entire trajectory, and to set up ratios accordingly. In this manner the need for additional range-table columns is avoided. For example, let us assume that in the case of the same trajectory for which we have already found the value of Column 12 (art. 1008),

it has been determined that the effect of a 10% variation in atmospheric density operating only on those portions of the trajectory which are included within the limits of the zone 0-600 feet, amounts to only .04 of the effect a 10% variation operating throughout the entire trajectory. The amount of the latter being the value tabulated in Column 12, in this case 1292 yards, the effect of the 10% variation which is confined to the zone 0-600 feet then is  $.04 \times 1292 = 52$  yards.

1011. The ratios referred to above are called *air-density weighting factors*, since they express the weights of the effects of density variations confined within specified limits of the trajectory, relatively to the effects of equal variations extending to the entire trajectory. The values of the weighting factors for a given trajectory evidently depend upon the characteristics of the trajectory itself, as well as upon the sizes of the zones to which the factors are to apply. However, the establishment and use of a separate series of such factors, for each of the great many trajectories included in the range tables of the various guns that are in use, would involve a degree of elaboration that is hardly warranted by the practical limitations of accuracy that apply to the measurement of aloft densities themselves. It is the present practice, therefore, to use for surface fire a single table of air-density weighting factors, which is based on a standard series of altitude zones and on a classification of trajectories only as to the number of such zones included within the limits of their respective maximum ordinates. The values of the weighting factors contained in this table represent averages deduced from a comprehensive analysis of such factors for all trajectories (i.e., including all guns) that are to be served by the table. Despite the wide limits assumed in obtaining these average factors, the latter afford a degree of accuracy that is commensurate with the accuracy of the density determinations themselves.

1012. The following table of air-density weighting factors is based on an analysis made at the Aberdeen Proving Ground according to the principles outlined above.\* The arguments to be used in entering this table are the maximum ordinate of the trajectory (vertical argument), and the limits of altitude (horizontal argument) to which the weighting factor applies. It will be observed that the horizontal argument represents zones of altitude; these zones have been accepted as standard in connection with the measurement of aloft densities. The arrangement and use of the table can best be explained by examples.

Let us assume a trajectory whose maximum ordinate is 9,000 feet. We find

---

\* This table is based on the formula

$$1 - p = 0.48 (1 - k)^{1/2} + 0.52 (1 - k)^{3/2}$$

in which  $p$  denotes the weighting factor, and  $k$  denotes the ratio  $y/y_s$  to which the weighting factor pertains. For example, to find the weighting factor for the portions of the trajectory included between the surface and a height equal to one-half of the maximum ordinate, the value  $k = \frac{1}{2}$  is substituted in the formula and the value of  $p$  is found to be .48. To find the weighting factor for the portions of a trajectory included between heights equal, respectively, to one-half and three-quarters that of the maximum ordinate, we find the difference between the values of  $p$  corresponding, respectively, to  $k = \frac{3}{4}$  and  $k = \frac{1}{2}$ , which results, in this case, in the value  $p = .69 - .48 = .21$ . Further details relative to the derivation of these weighting factors are given in Chapter XV, *A Course in Exterior Ballistics* (War Department Document No. 1051, December, 1920), by R. S. Hoar, and on pp. 237-243, *Computation of Firing Tables for the U. S. Army*, by H. P. Hitchcock.

The U. S. Army now uses two sets of air-density weighting factors, one for ordinary terrestrial fire and one for antiaircraft and other high-angle fire. These weighting factors, and the corresponding ballistic-density tables, are given in Tables IX to XII, *Technical Regulations No. 1236-1* (U. S. War Department, June, 1934).

## Air-density Weighting Factors

Maximum ordinate (feet)	Zones (feet)										
	0-600	600-1,500	1,500-3,000	3,000-4,500	4,500-6,000	6,000-9,000	9,000-12,000	12,000-15,000	15,000-18,000	18,000-24,000	24,000-30,000
600	1.00	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
1,500	.39	0.61	.....	.....	.....	.....	.....	.....	.....	.....	.....
3,000	.20	.28	0.52	.....	.....	.....	.....	.....	.....	.....	.....
4,500	.13	.19	.31	0.37	.....	.....	.....	.....	.....	.....	.....
6,000	.10	.14	.24	.21	0.31	.....	.....	.....	.....	.....	.....
9,000	.06	.10	.16	.16	.15	0.37	.....	.....	.....	.....	.....
12,000	.05	.07	.12	.12	.12	.21	0.31	.....	.....	.....	.....
15,000	.04	.06	.10	.10	.09	.18	.16	0.27	.....	.....	.....
18,000	.04	.04	.08	.08	.08	.16	.15	.14	0.23	.....	.....
24,000	.03	.03	.06	.06	.06	.12	.12	.11	.10	0.31	.....
30,000	.02	.03	.05	.05	.05	.10	.09	.09	.09	.16	0.27

tabulated against the vertical argument 9,000 and the horizontal argument 0-600, the value .06. This means that for a trajectory whose maximum ordinate is 9,000 feet, the weighting factor for the portions of the trajectory included between the surface and a height of 600 feet, is .06; and further, that for this trajectory the change in range due to a 10% variation in atmospheric density which is confined to the zone 0-600 feet, is equal to .06 of the change in range due to a 10% variation in density that extends throughout the entire trajectory (i.e., .06 of the value of Column 12 for that trajectory). Similarly, against the vertical argument 9,000 and the horizontal argument 6,000-9,000, we find the weighting factor .37, which means that, for the same trajectory, the change in range due to a 10% variation in atmospheric density which is confined to the zone 6,000-9,000 feet, is equal to .37 of the value of Column 12 for that trajectory. The complete series of weighting factors for the same trajectory is as given in the following table.

<u>Zone (feet)</u>	<u>Weighting factor</u>	<u>Change in range due to 10% variation in air density (yds.)</u>
0-600	.06	60
600-1,500	.10	100
1,500-3,000	.16	160
3,000-4,500	.16	160
4,500-6,000	.15	150
6,000-9,000	.37	370
	<u>1.00</u>	<u>1000</u>

Assuming that the value of Column 12 for this trajectory is 1000 yards, the changes in range due to a 10% variation in air density confined to the several zones included within the trajectory, are as shown in the right-hand column of the above table. By the same process, we can apply a *different* density variation in each zone and find the corresponding change in range for each zone, in which case the sum of these changes represents the aggregate effect on the whole trajectory of the several different density variations, each duly weighted according to its own zone.

1013. In practice it is more convenient to apply the weighting factors directly to the density factors of the several zones, and thus to deduce a weighted mean density factor, or *ballistic density*, for the entire trajectory (art. 421). For example, let us assume density factors for the various zones as shown in the following table; it is to be understood that each of these density factors is the ratio of the actual density

Use of air-density weighting factors for determining ballistic density

observed in that zone to the standard density *for the same zone*. (The values given are not to be considered as typical; round numbers have been chosen for convenience in following this elementary example.) Let us also assume the same trajectory for which the weighting factors have already been found in the foregoing article, and find the ballistic density for this trajectory.

<u>Zone (feet)</u>	<u>Observed density factor</u>	<u>Weighting factor</u>	<u>Weighted density factor</u>
0-600	1.060	.06	.0636
600-1,500	1.050	.10	.1050
1,500-3,000	1.040	.16	.1664
3,000-4,500	1.030	.16	.1648
4,500-6,000	1.020	.15	.1530
6,000-9,000	1.010	.37	.3737
			<u>1.0265</u>

The ballistic density for this trajectory is therefore 1.026, or, in other words, the weighted mean variation in atmospheric density for the entire trajectory, with respect to standard, is (+) 2.6%. The value of Column 12 for this trajectory being 1000 yards, the change in range for the whole trajectory, due to the aggregate effect of the several density variations, then is  $(-)\frac{2.6}{10}\times 1000 = (-) 260$  yards.

The principal purpose of the foregoing demonstrations of the methods by which air-density weighting factors are established, and by which the latter are used to determine a ballistic density, is to show that Column 12 of the range table is to be used in connection with a ballistic density just as with any other density. In actual practice, the preparation of ballistic densities from actual aloft observations is usually accomplished by aerological parties (see note adjacent to art. 423); or, if aloft observations are not available, the ballistic density is found from Table IV, as already explained in article 424. In either case the ballistic density embodies the air-density weighting factors.

1014. For further convenience in connection with the use of Column 12 of the range table, a table of multipliers for this column is given in Table V, *Range and Ballistic Tables, 1925*. These multipliers are simply the ratios between any given variation in atmospheric density and the tabular variation of 10% on which Column 12 is based. For example, for a 10% variation the multiplier is 1.00, for a 5% variation it is .50, for a 1% variation .10, etc. Signs are appended to these multipliers, to indicate the direction of the change of range that corresponds to the density variation to which the multiplier applies. The arguments for Table V are the surface density factor  $\delta$ , and the maximum ordinate. For zero height (i.e., zero maximum ordinate), the values given in the table are the multipliers corresponding to the surface density, and they correspond to the relation

$$M = \frac{1 - \delta}{.10} \quad (1001)$$

For any other height, the multipliers correspond to the ballistic density for a trajectory having a maximum ordinate of that height, and the relation is

$$M = \frac{1 - \delta_b}{.10} \quad (1002)$$

The following example will illustrate this further.

*Given:* Surface temperature 52°F., and barometer 30."40.

*Find:* Compute the multiplier for Column 12 of the range table, (a) for surface density, and (b) for ballistic density corresponding to a maximum ordinate of 10,000 feet.

From Table III we find  $\delta = 1.045$ , whence, from (1001) we find

$$M = \frac{1 - 1.045}{.10} = \underline{(-) .45.}$$

Entering Table IV with surface density 1.045 and maximum ordinate 10,000 feet, we find  $\delta_b = 1.035$ , whence

$$M = \frac{1 - 1.035}{.10} = \underline{(-) .35.}$$

Both of these results can be obtained directly from Table V.

1015. The multipliers for Column 12 are applicable directly to the values in that column. For example, if the value of the multiplier is  $(-).35$ , it means that the change in range for the conditions represented by that multiplier is .35 times the value given in Column 12, and the minus sign indicates the change is a shortening of the range; a plus sign indicates an increase in range.

It is to be noted that if Table V is entered with a ballistic density, the corresponding multiplier must be taken from the line for zero height. This is so because the values in the body of Table V represent the operations both of reducing a surface density factor to the ballistic density, and of deriving from the latter the corresponding multiplier. Therefore if the density factor used in entering this table is already a ballistic density, the only operation remaining to be performed is that indicated by (1002), and since the latter is identical in form with (1001), the required operation is represented in the multiplier for zero height. The practical applications of Column 12, and of the multipliers therefor, will be dealt with further in Chapter 12.

Determination of the multiplier corresponding to a ballistic density

### CHANGE IN RANGE DUE TO A GIVEN VARIATION IN WEIGHT OF PROJECTILE (COLUMN 11).

1016. In U. S. Navy range tables, Column 11 gives the change in range due to a small increase or decrease in the weight of the projectile. The variation in weight assumed for this column depends on the caliber of the gun, and is stated in the heading of the column. The heading of Column 11 in the more recent tables states that the values in this column are the changes in range corresponding to a minus variation (i.e., decrease) in the weight of the projectile. This is not intended to imply that the values were *derived* on the basis of a minus variation only, but merely to indicate that the sign of the tabulated values corresponds to a decrease in weight (the plus sign is to be understood when no sign appears before the tabulated values). A decrease in weight of the projectile generally results in an increase in range, but not necessarily so. It will be observed, for example, that in the case of the 5"3150 f.s. gun, a decrease in weight of projectile *increases* the range for ranges less than

Size and sign of variations assumed for Column 11

8000 yards, and *decreases* the range for ranges greater than 8000 yards. The reason for this will appear presently.

1017. The change in range due to a variation in the weight of the projectile is the resultant of two distinct effects, viz., (1) a change in the initial velocity, and (2) a change in the ballistic coefficient. An increase in the weight of the projectile operates to decrease the range due to a decrease in the initial velocity, and to increase the range due to an increase in the ballistic coefficient. For a decrease in weight the same effects occur with reversed signs. In any case, the two effects oppose each other, and they may even cancel each other (as is seen to be the case, for example, in the 5"3150 f.s. range table at 8000 yards). We shall denote a variation in the weight of the projectile by  $\Delta w$ , the corresponding variations in initial velocity and ballistic coefficient, respectively, by  $\Delta V_w$  and  $\Delta C_w$ , and the change in range which is the resultant of both of these effects, by  $\Delta X_w$ .

The determination of  $\Delta V_w$  is a problem in interior ballistics; it may be handled either by differentiating the velocity formula with respect to  $w$  alone and thus obtaining a relation between  $\Delta V_w$  and  $\Delta w$ , or by the direct process of solving the velocity formula for any assumed increased and decreased weights of projectile, whence the mean of the values of  $\Delta V_w$  corresponding to plus and minus values of  $\Delta w$  of any given magnitude may be obtained.\* Variations in projectile weight are held within narrow limits in manufacture, and we need concern ourselves, therefore, only with relatively small values of  $\Delta w$ . Changes in initial velocity corresponding to small variations in weight of projectile are given with sufficient accuracy by the formula

$$\Delta V_w = (-)m \times \frac{\Delta w}{w} \times V \quad (1003)$$

in which  $m$  is a coefficient whose value varies from about .20 to about .40. The value  $m = .36$  has been used for most of our range tables, although for some of the smaller guns smaller values of  $m$  have recently been found more satisfactory (for example,  $m = .26$  has been adopted for a recent 4" range table). The minus sign before  $m$  is accounted for by the fact that a minus velocity variation corresponds to a plus weight variation, and a plus velocity variation to a minus weight variation.

It is evident, from formula (406), that  $\Delta C_w$  is directly proportional to  $\Delta w$ . Hence we may write

$$\frac{\Delta C_w}{C} = \frac{\Delta w}{w}.$$

Change in ballistic coefficient due to change in weight of projectile

A more convenient form of the above relation, for our present purpose, is

$$\Delta C_w = \frac{\Delta w}{w} \times 100 \quad (1004)$$

in which  $\Delta C_w$  is expressed directly as a percentage variation with respect to  $C$ .

1018. Having determined the changes in initial velocity and ballistic coefficient due to a given change in weight of projectile, it is a simple matter to find the corresponding changes in range from Columns 10 and 12, respectively. (It is to be noted that Column 12, although designed to be used primarily in connection

\* The necessary formulas, and examples of solution, are given in Section V, Chapter III, *Naval Ordnance, 1933*.

with variations in atmospheric density, may be used also in connection with variations in any other factors contained in  $C$ . The sign to be used with values from this column depends, of course, on whether the factor to be dealt with is in the numerator or denominator of the formula for  $C$  (406).

The following example illustrates the determination of values for Column 11 according to the principles that have been outlined above.

*Given:* In the computation of range-table values for the 16"2600 f.s. gun, for  $\phi = 25^\circ$ , the values for Columns 10 and 12 have been found to be, respectively, 180 yards and 1292 yards. The standard weight of projectile is 2100 lbs., and the value of the coefficient  $m$  is .36.

*Find:* The change in range due a variation of  $\pm 10$  lbs. in weight of projectile.

From (1003) we have

$$\Delta V_w = (-) .36 \times \frac{(\pm) 10}{2100} \times 2600 = (\mp) 4.46 \text{ f.s.}$$

whence from Column 10 we have

$$\Delta X_1 = \frac{(\mp) 4.46}{10} \times 180 = \underline{(\mp) 80 \text{ yards.}}$$

From (1004) we have

$$\Delta C_w = \frac{(\pm) 10}{2100} \times 100 = (\pm) 0.48\%$$

whence from Column 12 we have

$$\Delta X_2 = \frac{(\pm) .48}{10} \times 1292 = \underline{(\pm) 62 \text{ yards.}}$$

Combining the two changes,  $\Delta X_1$  and  $\Delta X_2$ , we have, finally,

$$\Delta X_w = (\mp) 80 (\pm) 62 = \underline{(\mp) 18 \text{ yards.}}$$

The result we have found is that an increase of 10 lbs. in  $w$  decreases the range 18 yards, and a decrease of 10 lbs. in  $w$  increases the range 18 yards.\*

1019. As a further example, let us take the following.

*Given:* For the 5"3150 f.s. gun, the following values have been found (all in yards).

<u>Range</u>	<u>Col. 10</u>	<u>Col. 12</u>
1000	6	6
8000	31	352
12000	39	643

The standard weight of projectile is 50 lbs., and the value of the coefficient  $m$  is .36.

*Find:* The change in range due to a variation of  $\pm 1$  lb. in weight of projectile (Col. 11) at each of the given ranges.

Proceeding just as in the first example, we have

$$\Delta V_w = (-) .36 \times \frac{(\pm) 1}{50} \times 3150 = (\mp) 22.68 \text{ f.s.}$$

\* See note on page 127.

$$\Delta C_w = \frac{(\pm) 1}{50} \times 100 = (\pm) 2\%$$

and these values apply at all ranges. For each of the given ranges the solution is then completed as follows:

For range 1000 yards

$$\Delta X_1 = \frac{(\mp) 22.68}{10} \times 6 = \underline{(\mp) 14 \text{ yards}}$$

$$\Delta X_2 = \frac{(\pm) 2}{10} \times 6 = \underline{(\pm) 1 \text{ yard}}$$

$$\Delta X_w = (\mp) 14 (\pm) 1 = \underline{(\mp) 13 \text{ yards}}$$

For range 8000 yards

$$\Delta X_1 = \frac{(\mp) 22.68}{10} \times 31 = \underline{(\mp) 70 \text{ yards}}$$

$$\Delta X_2 = \frac{(\pm) 2}{10} \times 352 = \underline{(\pm) 70 \text{ yards}}$$

$$w = (\mp) 70 (\pm) 70 = 0$$

For range 12,000 yards

$$\Delta X_1 = \frac{(\mp) 22.68}{10} \times 39 = \underline{(\mp) 88 \text{ yards}}$$

$$\Delta X_2 = \frac{(\pm) 2}{10} \times 643 = \underline{(\pm) 129 \text{ yards}}$$

$$\Delta X_w = (\mp) 88 (\pm) 129 = \underline{(\pm) 41 \text{ yards}}$$

This example shows that, for the 5"3150 f.s. gun, a decrease in weight of projectile results in an increase of range at ranges less than 8000 yards, and a decrease of range at ranges greater than 8000 yards. This accounts for the change of sign that occurs in Column 11 of this range table at 8000 yards. It is of interest to note that at 8000 yards a change in weight of projectile has practically no effect on the range. In the case of the 16"2600 f.s. gun, for similar reasons, the value in Column 11 reaches a maximum at the range of about 15,000 yards, and beyond that it decreases, although it does not reach the zero value within the limits of the table.

### EXERCISES

1. *Given:* The initial velocity, diameter, weight, and coefficient of form of the projectile, and the angle of departure.

*Find:* The change in range due to a variation of  $(-)$  10 f.s. in initial velocity. (Use A.L.V.F. Tables, and use  $V$  to nearest one-tenth of a m.s.).

	Given					Answers
	$V$ (f.s.)	$d$ (in.)	$w$ (lbs.)	$i$	$\phi$	$\Delta X_V^*$ (yards)
A	2600	16	2100	.61230	15°	(-)138
B	2600	16	2100	.61200	30	(-)205
C	2600	16	2100	.61140	40	(-)241

\* See note on page 127.

2. *Given:* The initial velocity, diameter, weight, and coefficient of form of the projectile, and the angle of departure.

*Find:* The change in range due to a variation of ( $\pm$ ) 10% in atmospheric density. (Use A.L.V.F. Tables.)

	Given					Answers
	$V$ (f.s.)	$d$ (in.)	$w$ (lbs.)	$i$	$\phi$	$\Delta X_C^*$ (yards)
A	2600	16	2100	.61230	15°	( $\mp$ ) 778
B	2600	16	2100	.61200	30	( $\mp$ )1498
C	2600	16	2100	.61140	40	( $\mp$ )1752

\* See note on page 127.

3. *Given:* The range, initial velocity, weight of projectile, coefficient  $m$ , and the values from Columns 10 and 12 of the range table (16"2600 f.s. gun).

*Find:* The change in range due to the stated variation in weight of projectile, for the given range.

	Given							Answers
	$X$ (yards)	$V$ (f.s.)	$w$ (lbs.)	$\Delta w$ (lbs.)	$m$	Col. 10 (yards)	Col. 12 (yards)	$\Delta X_w^*$ (yards)
A	3000	2600	2100	( $\pm$ )10	.36	22	16	( $\mp$ ) 9
B	8000	2600	2100	( $\pm$ )10	.36	55	107	( $\mp$ )20
C	20000	2600	2100	( $\pm$ )10	.36	115	594	( $\mp$ )22

\* See note on page 127.

THE DETERMINATION OF THE EFFECTS OF WIND AND MOTION OF GUN ON THE TRAJECTORY, AND OF MOTION OF TARGET ON THE POINT OF FALL WITH RESPECT TO THE TARGET (RANGE-TABLE COLUMNS 13-18).

New Symbols Introduced

- $W_x, G_x, T_x$  . . . . . Components of wind, gun motion, and target motion, respectively, in the line of fire.
- $W_z, G_z, T_z$  . . . . . Components of wind, gun motion, and target motion, respectively, perpendicular to the line of fire.
- $\Delta\phi_w, \Delta\phi_g$  . . . . . Apparent changes in  $\phi$  due to wind and gun motion, respectively, in the line of fire.
- $\Delta V_w, \Delta V_g$  . . . . . Apparent changes in  $V$  due to wind and gun motion, respectively, in the line of fire.
- $\Delta X_\phi$  . . . . . The change in range corresponding to a change in  $\phi$  (as for example  $\Delta\phi_w$  or  $\Delta\phi_g$ ).
- $\Delta X_w, \Delta X_g, \Delta X_t$  . . . . Changes in range due to components of wind, gun motion, and target motion, respectively, in the line of fire.
- $D_w, D_g, D_t$  . . . . . Lateral deviations due to components of wind, gun motion, and target motion, respectively, perpendicular to the line of fire.

1101. Columns 13-18 of the range table deal with the effects of wind, motion of gun, and motion of target, all of which are assumed to occur in a horizontal plane through the origin or in a plane parallel thereto. Columns 13, 14, and 15, deal, respectively, with the effects of components of wind, motion of gun, and motion of target occurring wholly *in* the line of fire, and Columns 16, 17, and 18, respectively, with the effects of such components occurring wholly *perpendicular* to the line of fire (all in the horizontal plane).

It was formerly the practice to base Columns 13-18 on 12-knot components, and a few range tables based on this practice are still in use. The present practice is to base these columns on 10-knot components. The size of the component on which these columns are based is always stated in the headings of the columns. No signs appear in these columns, but the proper sign can always be inferred from the sign of the component itself; this matter will be dealt with further in Chapter 12. All of the examples and exercises given in this text will be based on 10-knot components, and on the relation 1 sea mile = 6080 feet, whence 1 knot = 1.689 foot-seconds.

It will be convenient to take up first the effects of target motion, Columns 15 and 18, since the values for these columns can be used to advantage in the computation of the other columns.

CHANGE IN RANGE DUE TO A GIVEN COMPONENT OF TARGET MOTION IN THE LINE OF FIRE (COLUMN 15).

1102. Motion of the target in no way affects the trajectory itself, but nevertheless it affects the position of the point of fall with respect to the target, and

hence its result is the equivalent of a disturbance of the trajectory itself. We evidently must lay the gun in elevation, not for the range to the target at the instant of firing, but for this range plus or minus the change in range that ensues from the target's motion during the time of flight.

We shall denote by  $T_x$  the component of target motion in the line of fire, and by  $\Delta X_T$  the change in range resulting therefrom. The latter amounts to the product of the target motion and the time of flight, or

$$\Delta X_T = T_x \times T. \tag{1101}$$

Strictly speaking, the value of  $T$  used in (1101) should be the time of flight corresponding to the corrected range,  $X \pm \Delta X_T$ , rather than the time of flight for the assumed range  $X$  (i.e., the range to the target at the instant of firing). If this distinction is to be made in the value of  $T$ , then (1101) must be solved by successive approximations, the first approximation being based on the value of  $T$  which corresponds to the assumed range  $X$ ; further approximations can then be based on values of  $T$  corresponding to the corrected ranges  $X \pm \Delta X_T$ . The values given in Column 15 are the first approximations of  $\Delta X_T$ , and hence are slightly too great for target motion toward the gun and slightly too small for target motion away from the gun. If necessary, closer approximations can be derived from the tabular values, without resorting to additional computations, as will be illustrated presently.

1103. The computation of values for Column 15 is illustrated in the following example.

*Given:* The 16" 2600 f.s. gun, range 31,200 yards, for which the time of flight equals 57.79 seconds.

Computation  
of Column 15

*Find:* The change in range due to a 10-knot component of target motion in the line of fire (Column 15).

Applying formula (1101), we have

$T_x = 10$ knots.....	log 1.00000
1.689 (art. 1101).....	log 0.22763
$T = 57.79$ seconds.....	log 1.76185
$\Delta X_T = 976.08$ feet.....	log 2.98948
<u>= 325 yards</u>	

The result thus obtained means that if the sight-bar range of 31,200 yards will cause a hit on a motionless target, it will cause a miss 325 yards short of a target moving at ten knots directly away from the gun, and a miss 325 yards beyond a target moving at ten knots directly toward the gun. How-

Accuracy  
of Column 15

ever, if we change the sight-bar range to  $31,200 + 325 = 31,525$  yards for the target moving away, the point of fall will still occur

slightly short of the target, due to the increase in time of flight for the increased sight-bar range. From Column 15 we find that during the time of flight corresponding to the range 31,525 yards, the motion of the target is 330 yards, and hence a more accurate determination of the sight-bar range required to intercept the target is  $31,200 + 330 = 31,530$  yards. Further approximations will not change the latter result. Proceeding similarly with the case of a target moving at 10 knots directly toward the gun, we find as a first approximation of the corrected sight-bar range  $31,200 - 325 = 30,875$  yards, and as a second (and final) approximation  $31,200 - 320 = 30,880$  yards. In either of the above cases the difference between  $\Delta X_T$  as found directly from the range table and as found by successive approxi-

mations, is of no practical consequence, especially in comparison with the errors that are likely to result from inaccurate knowledge of the target speed itself. This remains true even for the greatest target speeds that are likely to be encountered in surface fire. It is the usual practice, therefore, to use the values from Column 15 as they are given in the range table (i.e., without resorting to the further approximations outlined above).

#### LATERAL DEVIATION DUE TO A GIVEN COMPONENT OF TARGET MOTION PERPENDICULAR TO THE LINE OF FIRE (COLUMN 18).

1104. The computation of values for Column 18 is identical with that for Column 15. We shall denote by  $T_z$  the component of target motion perpendicular to the line of fire, and by  $D_T$  the lateral deviation resulting therefrom. Then we have

$$D_T = T_z \times T \quad (1102)$$

in which  $T$  is the time of flight corresponding to the given range. Since the range is not appreciably affected by  $D_T$ , the time of flight also is not appreciably affected, and a direct solution of (1102) gives as great a degree of accuracy as is required for any purpose. The computation for Column 18 is illustrated in the following example.

*Given:* The 16" 2600 f.s. gun, range 31,200 yards, for which the time of flight equals 57.79 seconds.  
*Find:* The lateral deviation due to a 10-knot component of target motion perpendicular to the line of fire (Column 18).

Applying formula (1102), we have

$T_z = 10$ knots . . . . .	log 1.00000
1.689 (art. 1101) . . . . .	log 0.22763
$T = 57.79$ . . . . .	log 1.76185
$D_T = 976.08$ feet . . . . .	log 2.98948
<u><math>= 325</math> yards</u>	

The result thus obtained means that, in this case, the gun must be directed 325 yards to the right if the target motion is to the right, and 325 yards to the left if the target motion is to the left, in order that the point of fall may occur on the moving target. The method by which the gun is offset to counteract lateral deviations will be explained in Chapter 12.

#### CHANGE IN RANGE DUE TO A GIVEN WIND COMPONENT IN THE LINE OF FIRE (COLUMN 13).\*

1105. Although it is well known that the wind may at times have vertical components of appreciable magnitude, practical methods of measuring the wind are limited to the determination of its horizontal components, and provisions for the practical determination of the effects of wind on the trajectory are accordingly limited to horizontal winds. Lack of knowledge of vertical wind components is, however, not the only limitation involved in our determination of the effects

\* See also Appendix D.

Limitations  
involved  
in correcting  
for effects  
of wind

of the wind, for even in the measurement of the horizontal wind we are limited to a relatively low order of approximation, because of the practical difficulties involved in obtaining aloft observations that are immediately applicable to the time of firing. Moreover, actual aloft observations may often be lacking altogether, in which case the determination of aloft winds rests entirely on estimate. It is to be appreciated, therefore, that the practical determination of the effects of wind on the trajectory may often be largely a matter of speculation.

Range wind  
and cross  
wind

The range-table columns which deal with the effects of wind on the trajectory are based on a horizontal wind which is considered to be of uniform magnitude and direction throughout the trajectory. Column 13 is based on a component of such wind measured *in* the line of fire, and Column 16 on a component measured *perpendicularly* to the line of fire. We shall refer to these components, respectively, as *range wind* and *cross wind*.

General  
principles  
governing  
determination  
of effects  
of wind

1106. In analyzing the effect of wind on the flight of the projectile, it is convenient to consider separately the motion of the projectile with respect to the moving air, and the motion of the air itself. If the air is in motion away from the gun, we may consider, then, that the initial velocity of the projectile with respect to this moving air is less than it would be with respect to still air, and that the projectile consequently travels a shorter distance through the moving air itself than it would travel through the still air itself. However, while the projectile in this case travels a shorter distance with respect to the air, the latter itself moves bodily forward, and the travel of the projectile with respect to the ground is the resultant of its travel through the air and of the latter's motion over the ground. Similar reasoning applies to the case of air moving toward the gun; in this case the projectile travels a greater distance with respect to the air, but the latter itself moves backward. The loss or gain of distance with respect to the moving air is always less than the accompanying gain or loss of distance due to the motion of the air over the ground, whence it follows, as is to be expected, that a wind blowing away from the gun increases the range and a wind blowing toward the gun decreases the range.

Let us denote by  $W_x$  the range wind (positive for wind blowing away from the gun, and negative for wind blowing toward the gun), and by  $\Delta X_w$  the change in range resulting therefrom. Then if  $X_1$  represents the range with respect to the moving air, as compared to the range  $X$  in still air, we may write

$$\Delta X_w = X_1 \pm W_x T - X. \quad (1103)$$

In the above formula, the range over the ground, under the condition of a wind  $W_x$ , is represented by  $X_1 \pm W_x T$ ; for  $X_1$  represents the range with respect to the moving air, and  $W_x T$  the distance through which the air itself moves. Strictly speaking,  $T$  should be the time of flight for the range  $X_1$ , but no material error is occasioned by taking it as the time of flight corresponding to the range  $X$ . (The situation here is comparable to that already discussed in article 1102.)

1107. The above analysis of the effect of a range wind is illustrated graphi-

cally in Figure 22, which represents the case of a wind blowing away from the gun. In the upper diagram,  $OSH$  represents the trajectory that results from given values of  $\phi$  and  $V$  when no wind is blowing, the range in this case being  $X = OH$ ;  $OS'H'$  represents the trajectory that results from the same  $\phi$  and  $V$  when a wind

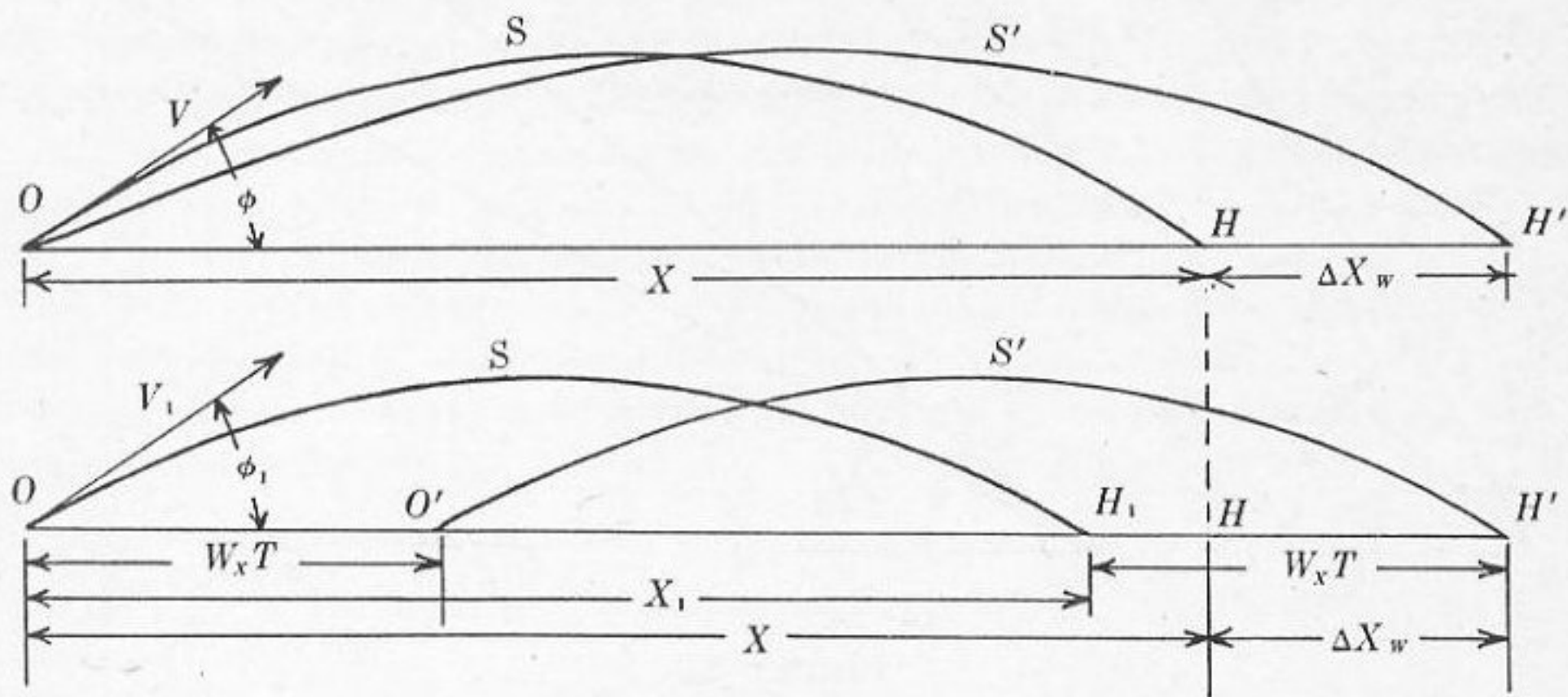


FIGURE 22

Graphical representation of effects of range wind

$W_x$  is blowing away from the gun, the range now being increased by  $\Delta X_w = HH'$ . In the lower diagram, the effect of the wind in increasing the range is analyzed into the two effects already discussed;  $OSH_1$  is the trajectory with respect to the moving air, and  $O'S'H'$  is the same trajectory translated forward through the distance  $W_x T$ . It is to be understood that the same *actual*  $\phi$  and  $V$  are considered to apply in the lower diagram as in the upper diagram, although the trajectory  $OSH_1$  of the lower diagram, and its corresponding range  $X_1 = OH_1$ , are assumed to result from an *apparent* angle of departure  $\phi_1$  and initial velocity  $V_1$ . The values  $\phi_1$ ,  $V_1$ , and  $X_1$  of the lower diagram are all to be understood as being measured in relation to the moving air. Figure 22 shows that  $\Delta X_w = HH'$  is the resultant of a decrease  $HH_1$  in range which the projectile suffers with respect to the moving air, and an increase  $W_x T = H_1 H'$  in range which it gains due to the bodily movement of the air itself.

1108. In order to solve formula (1103), we must determine the value of  $X_1$ , i.e., the range of the trajectory with respect to the moving air. This can be done by finding the values of the angle of departure and initial velocity with respect to the moving air; we shall denote these by  $\phi_1$  and  $V_1$ . In Figure 23 (a) the actual initial velocity is represented by the vector  $V = OA$ , making with the horizontal the angle  $AOH$  which is equal to the actual angle of departure  $\phi$ ; in other words,  $OA$  represents vectorially the initial motion imparted to the projectile with respect to the ground. In the same figure,  $AB$  represents vectorially a horizontal wind component  $W_x$  blowing away from the gun.  $OB$  then represents vectorially the initial motion of the projectile with respect to the moving air, and  $V_1 = OB$  and  $\phi_1 = BOH$  are the initial velocity and angle of departure with respect to the moving air. Figure 23 (b) similarly represents the case of a wind blowing toward

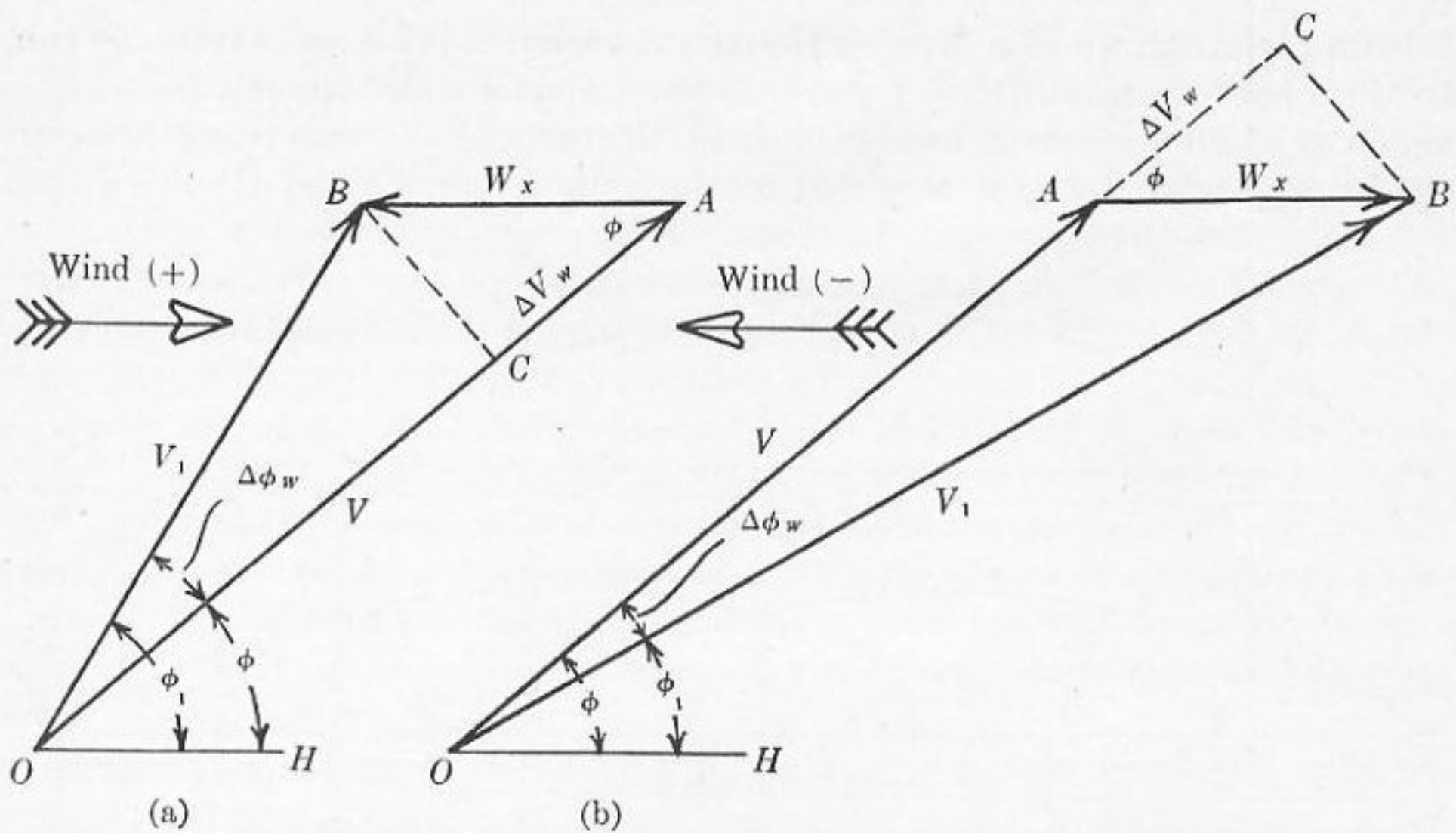


FIGURE 23

the gun. (The direction of the vector  $AB$  is correctly drawn in each case to show that  $OB$  is the resultant of  $OA$  and  $AB$ .)

It is seen that with a positive wind (Figure 23 (a)) there is an apparent increase in angle of departure and an apparent decrease in initial velocity, with respect to the moving air, while with a negative wind (Figure 23 (b)) there is an apparent decrease in angle of departure and an apparent increase in initial velocity, with respect to the moving air. We shall denote these apparent changes in angle of departure and initial velocity by  $\Delta\phi_w$  and  $\Delta V_w$ . If we drop a perpendicular from  $B$  to  $C$  in either of the above figures we have, in each case,  $OC$  practically equal to  $OB$  (since the angle  $AOB$  between the vectors  $OA$  and  $OB$  is always very small). Then  $AC$ , in each case, is practically equal to the difference in length between the two vectors  $OA$  and  $OB$ , and hence is practically equal to  $\Delta V_w$ , which represents the difference between  $V$  and  $V_1$ . From the construction of the figure we have, in either case,  $\Delta V_w = W_x \cos \phi$  (approximately). Also, in either case,

$$\sin \Delta\phi_w = \frac{BC}{OB}. \text{ But } BC = W_x \sin \phi, \text{ and } OB = V_1, \text{ whence } \sin \Delta\phi_w = \frac{W_x \sin \phi}{V_1}.$$

In any practical case, however, the difference between  $V_1$  and  $V$ , although sufficiently great to affect the range appreciably, is not great enough to affect the value of  $\Delta\phi_w$  appreciably. Therefore no material error is occasioned by using  $V$  in place of  $V_1$  to find  $\Delta\phi_w$  for both positive and negative winds.

1109. The apparent changes in  $V$  and  $\phi$  with respect to the moving air can therefore be found with sufficient approximation from the expressions

$$\Delta V_w = W_x \cos \phi \quad (1104)$$

$$\sin \Delta\phi_w = \frac{W_x \sin \phi}{V} \quad (1105)$$

and the apparent initial velocity and angle of departure resulting from these changes can then be found from the expressions

$$V_1 = V \pm \Delta V_w \quad (1106)$$

$$\phi_1 = \phi \pm \Delta\phi_w. \quad (1107)$$

With regard to signs, it is to be noted that the sign of  $\Delta\phi_W$  must always be contrary to that of  $\Delta V_W$ . In the case of a positive wind,  $\Delta\phi_W$  is positive and  $\Delta V_W$  negative; in the case of a negative wind,  $\Delta\phi_W$  is negative and  $\Delta V_W$  positive. Reference to Figures 23 (a) and (b) will make this clear.

The range corresponding to  $\phi_1$  and  $V_1$  is the range with respect to the moving air, or the value  $X_1$  appearing in formula (1103). Since  $\Delta\phi_W$  and  $\Delta V_W$  are always small, we can determine their effects on the range readily from data already computed for the range table, as follows. Let us denote by  $\Delta X_\phi$  the change in range corresponding to the change  $\Delta\phi_W$  in angle of departure, and by  $\Delta X_V$  the change in range corresponding to the change  $\Delta V_W$  in initial velocity. The value of  $\Delta X_\phi$  can then be found from Column 2 (b) of the range table,\* which gives the value of  $\Delta\phi$  corresponding to  $\Delta X = 100$  yards. The value of  $\Delta X_V$  can, of course, be found from Column 10. In accordance with the rule for signs given in the preceding paragraph, we can then write,

(a) for a positive wind (+ $W_X$ )

$$X_1 = X + \Delta X_\phi - \Delta X_V$$

whence from (1103)

$$\begin{aligned}\Delta X_W &= X + \Delta X_\phi - \Delta X_V + W_X T - X \\ &= + (W_X T + \Delta X_\phi - \Delta X_V)\end{aligned}$$

(b) for a negative wind (- $W_X$ )

$$X_1 = X - \Delta X_\phi + \Delta X_V$$

whence from (1103)

$$\begin{aligned}\Delta X_W &= X - \Delta X_\phi + \Delta X_V - W_X T - X \\ &= - (W_X T + \Delta X_\phi - \Delta X_V).\end{aligned}$$

It follows from the above that the change in range due to a wind component  $W_X$  in the line fire, can be found from the formula

Formula for  
determination  
of effect of  
range wind

$$\Delta X_W = W_X T + \Delta X_\phi - \Delta X_V \quad (1108)$$

in which  $\Delta X_W$  takes the same sign as the wind component itself, and in which  $T$  is the time of flight corresponding to the given range  $X$ ,  $\Delta X_\phi$  is the change in range due to a change  $\Delta\phi_W$  in angle of departure as found from (1105), and  $\Delta X_V$  is the change in range due to a change  $\Delta V_W$  in initial velocity as found from (1104). A further simplification of the above results from the fact that the term  $W_X T$  is identical in form with the formula  $T_X T$  (1101) from which Column 15 of the range table has already been computed. For a 10-knot component of wind ( $W_X = 10$  knots), the value of  $W_X T$  can therefore be taken directly from Column 15, which is also based on a 10-knot component. Formula (1108) can then be written finally

$$\Delta X_W = (\text{Col. 15}) + \Delta X_\phi - \Delta X_V. \quad (1109)$$

1110. The solution of (1109) is illustrated in the following example.

Computation  
of Column 13

*Given:* The 16"2600 f.s. gun,  $\phi = 25^\circ$ , and the following range-table values corresponding to this angle of departure: Col. 2 (b) = 8'.7, Col. 10 = 180 yards, Col. 15 = 325 yards.

*Find:* The change in range due to a wind component of 10 knots in the line of fire (Column 13).

\* Column 2 (b) does not appear in the earlier range tables, but the required data can always be found readily from Column 2.

Solving (1104) and (1105), we have

$W_x = 10$ knots.....	log 1.00000.....	log 1.00000
1.689 (art. 1101).....	log .22763.....	log .22763
$\phi = 25^\circ$ .....	lcos 9.95728-10.....	lsin 9.62595-10
$\Delta V_w = 15.3$ f.s.....	log 1.18491	
$V = 2600$ f.s.....		colog 6.58503-10
$\Delta\phi_w = 9' 5$ .....		lsin 7.43861-10

From Column 2 (b) we have

$$\Delta X_\phi = \frac{9.5}{8.7} \times 100 = \underline{109 \text{ yards}}$$

and from Column 10

$$\Delta X_v = \frac{15.3}{10} \times 180 = \underline{275 \text{ yards}}$$

whence from formula (1109)

$$\Delta X_w = 325 + 109 - 275 = \underline{159 \text{ yards}}.*$$

### LATERAL DEVIATION DUE TO A GIVEN WIND COMPONENT PERPENDICULAR TO THE LINE OF FIRE (COLUMN 16).†

1111. In analyzing the effect of a cross wind on the flight of the projectile, we shall again consider separately the motion of the projectile with respect to the moving air, and the motion of the air itself. If the air is in motion *from* the left, we may consider, then, that the projectile has an apparent motion *to* the left with respect to the air.‡ However, while the projectile is apparently moving to the left with respect to the air, the latter is itself moving bodily to the right. Similar reasoning applies to the case of air moving from the right. The projectile's lateral displacement to the left or right with respect to the moving air, is always less than the accompanying lateral displacement to the right or left of the air itself,

\* By a similar treatment, the effect of a vertical wind component  $W_y$  is found to be the resultant of a bodily raising or lowering of the entire trajectory by the amount  $W_y T$ , and of an apparent change in the trajectory with respect to the moving air. The change in range corresponding to the change in height  $W_y T$  can be found from Column 19; it is positive for a wind blowing upward, and negative for a wind blowing downward. From a vector analysis similar to that given in Figures 23 (a) and (b), it can be shown that for a vertical wind the ap-

parent changes in  $\phi$  and  $V$  with respect to the moving air, are  $\sin \Delta\phi_w = \frac{W_y \cos \phi}{V}$  and

$\Delta V_w = W_y \sin \phi$ , and that they are of like sign, both negative for a wind blowing upward, and both positive for a wind blowing downward. For a 10-knot vertical wind, in the case of the above example, the resultant change in range due to the operation of the three effects is  $443 - 232 - 128 = 83$  yards. An examination of the effect of vertical wind at other angles of departure (16"2600 f.s. gun) shows that the effect has its greatest value, about 120 yards for a 10-knot component, at about  $\phi = 15^\circ$ . For greater or lesser values of  $\phi$  it decreases, becoming about 100 yards at  $\phi = 5^\circ$ , and about 40 yards at  $\phi = 40^\circ$ . Although these values undoubtedly are only rough approximations, they serve to indicate that the degree of error occasioned by ignorance of vertical wind components is of less consequence than might be supposed, in long-range fire. On the other hand, it is worthy of note that at short ranges the effect of a vertical wind far exceeds that of a horizontal wind of equal magnitude.

† See also Appendix D.

‡ *Right* and *left* are to be considered as signifying directions referred to the direction of fire.

whence it follows, as is to be expected, that a wind blowing *from* the left causes a lateral deviation of the projectile *to* the right, and a wind blowing *from* the right a lateral deviation of the projectile *to* the left.

Let us denote by  $W_z$  the cross wind (positive for wind blowing *from* the left, and negative for wind blowing *from* the right), and by  $D_w$  the lateral deviation resulting therefrom. Then if  $D_1$  represents the lateral deviation with respect to the moving air, we may write

$$D_w = W_z T - D_1 \tag{1110}$$

which states simply that the lateral deviation with respect to the ground is the resultant of the lateral deviation with respect to the moving air and of the lateral displacement of the air itself during the time of flight.

1112. The above analysis of the effect of a cross wind is illustrated graphically in Figure 24 (a), which represents the case of a wind blowing from the left (i.e., a positive cross wind). Figure 24 (a) is a plan view in which  $OH$ ,  $OH_1$  and  $OH'$  represent the traces, on the horizontal plane, of the trajectories to which we shall refer in the following discussion.  $OH$  represents the trajectory unaffected by wind,  $OH_1$  the trajectory affected by the wind  $W_z$  and as described with respect to the moving air, and  $OH'$  the trajectory affected by the same wind  $W_z$  but as described with

Graphical representation of effects of cross wind

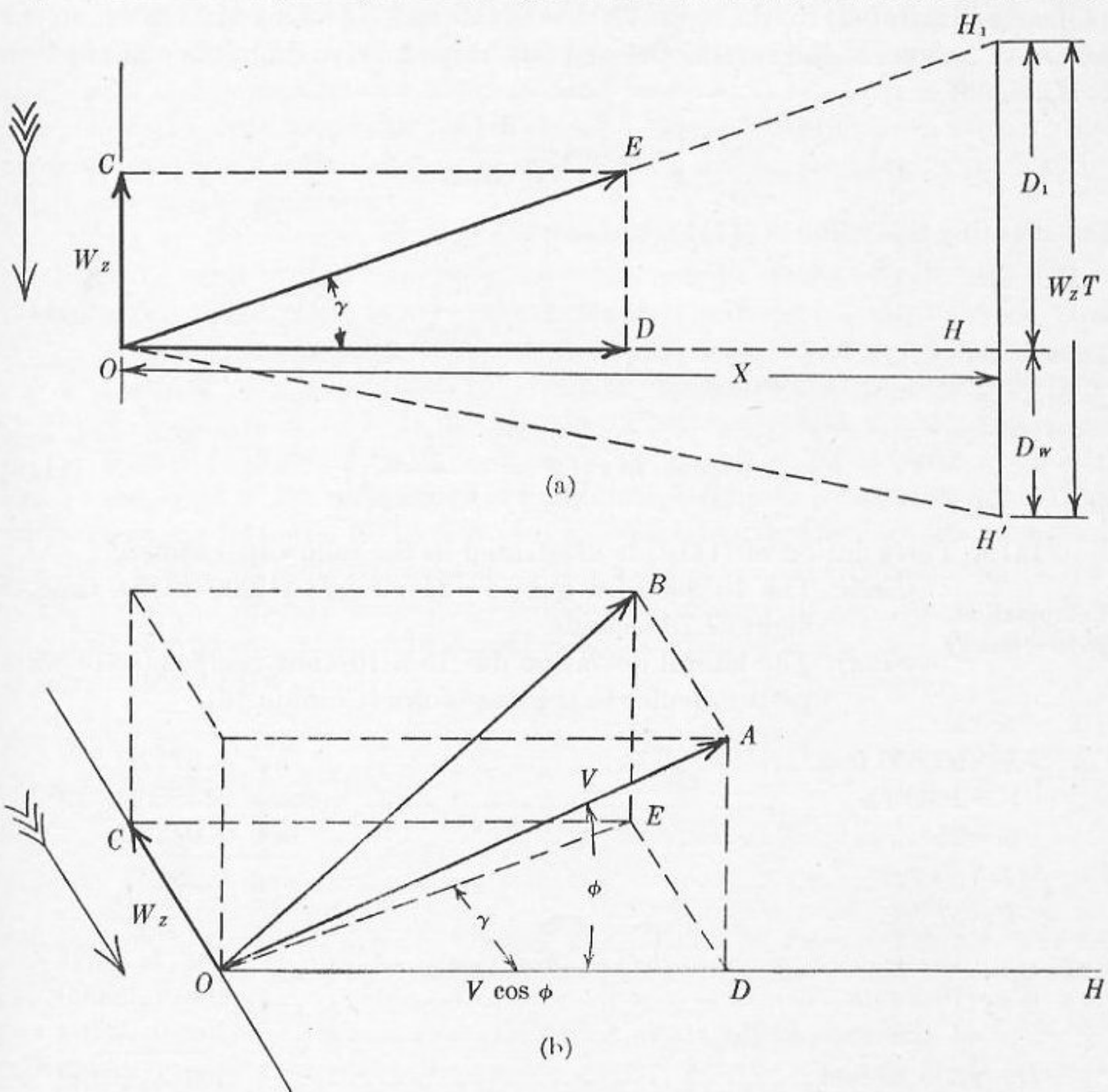


FIGURE 24

respect to the ground. The range is represented by  $X = OH$ . No distinction as to range need be made among  $OH$ ,  $OH_1$ , and  $OH'$ ; the differences among them, although exaggerated in the figure, are actually insignificant in any practical case. The lateral deviation with respect to the moving air is represented by  $D_1 = HH_1$ , or  $D_1 = X \tan \gamma$  in which  $\gamma$  is the angle  $DOE$ . The lateral displacement of the air during the time of flight is represented by  $H_1H' = W_z T$ . The resultant lateral deviation with respect to the ground is then  $HH' = D_w$ , and we have

$$D_w = W_z T - X \tan \gamma. \tag{1111}$$

The angle  $\gamma$  which is required for the solution of (1111) can be found from the vector analysis illustrated in Figure 24 (b), which is an elevation view, in perspective, corresponding to the plan view of Figure 24 (a). In Figure 24 (b) the actual initial velocity is represented by the vector  $V = OA$ , making with the horizontal the angle  $AOD = \phi$ , and the cross-wind is represented by the vector  $W_z = OC$ . The resultant of these two vectors is the vector  $OB$ , which represents the initial motion of the projectile with respect to the moving air. The vector  $OB$  represents a slight increase in initial velocity and decrease in angle of departure, as compared to  $OA$ , but these differences are too small to affect the range appreciably. The essential difference between the two vectors is the lateral angular displacement of  $OB$  with respect to  $OA$ , which is measured in the horizontal plane, as illustrated, by the angle  $DOE = \gamma$ ,  $OD$  and  $OE$  being the traces, on the horizontal plane, of the vectors  $OA$  and  $OB$ , respectively. From the construction it is evident that

$$\tan \gamma = \frac{W_z}{V \cos \phi}.$$

Substituting this value in (1111), we have

$$D_w = W_z T - \frac{X W_z}{V \cos \phi} \tag{1112}$$

Formula for determination of effect of cross wind or

$$D_w = W_z \left( T - \frac{X}{V \cos \phi} \right). \tag{1113}$$

1113. The solution of (1113) is illustrated in the following example.

Computation of Column 16

*Given:* The 16"2600 f.s. gun,  $\phi = 25^\circ$ , range 31,200 yards, time of flight 57.79 seconds.

*Find:* The lateral deviation due to a 10-knot component of wind perpendicular to the line of fire (Column 16).

$X = 93,600$ feet.....	log 4.97128
$V = 2600$ f.s.....	colog 6.58503 - 10
$\phi = 25^\circ$ .....	<u>lsec 0.04272</u>
(-) 39.72.....	log 1.59903
$T = 57.79$	
18.07.....	log 1.25696
$W_z = 10$ knots.....	log 1.00000
1.689 (art. 1101).....	<u>log 0.22763</u>
$D_w = 305.20$ feet.....	<u>log 2.48459</u>
= 102 yards	

## THE DETERMINATION OF THE EFFECTS OF WINDS WHICH ARE NOT UNIFORM THROUGHOUT THE TRAJECTORY.

1114. Except in the case of small angles of departure, it is to be expected that both the velocity and the direction of the wind will vary materially within the limits of the trajectory, for the velocity and direction of the wind ordinarily vary materially with the altitude. The problem of predicting the effects of such varying winds is quite similar to that of predicting the effects of the different density variations that occur within the limits of a trajectory, and it is handled in quite the same manner as the latter. The principles which govern the establishment of air-density weighting factors, as outlined in articles 1010-1011, are applied also in the establishment of wind weighting factors; the latter express the ratios of the effects of winds which are confined within given zones of altitude, to the effects of equal winds which operate uniformly throughout the entire trajectory. By applying the wind weighting factors for the several zones of a trajectory to the winds measured in these zones, a weighted mean wind, or *ballistic wind* is found. The ballistic wind is then evidently a fictitious uniform wind which is the equivalent of the several winds that actually exist within the limits of the trajectory. Wind weighting factors are not the same for range wind as for cross wind, and hence it follows that a theoretically correct determination of the range-wind component depends upon a ballistic wind based on range-wind weighting factors, and of the cross-wind component upon a ballistic wind based on cross-wind weighting factors; in other words, it may be said that two different ballistic winds exist for the same trajectory, one of which may be used only for finding the range wind and the other only for the cross wind.

However, the measurement of aloft winds is subject to practical limitations of about the same order as apply in the measurement of aloft air densities. It is logical practice, therefore, to avoid great elaboration in the establishment of wind weighting factors, as has already been done in the case of air-density weighting factors (art. 1011). A comprehensive analysis of wind weighting factors, made at the Aberdeen Proving Ground, has resulted in the establishment of a single formula, applicable either to range wind or to cross wind, from which sufficiently approximate weighting factors for any surface trajectory may be found.\* The following table is based on this formula, and gives the wind weighting factors for the same zones that have already been used in connection with air-density weighting factors. In arrangement and use, this table is quite similar to the table of air-density weighting factors that has already been given in article 1012.

\* The formula is

$$1 - p = 0.74 (1 - k)^{1/2} + 0.26 (1 - k)^2$$

in which  $p$  denotes the weighting factor, and  $k$  denotes the ratio  $y/y_s$  to which the weighting factor pertains (see also note adjacent to article 1012). Details entering into the derivation of wind weighting factors are given in Chapter XV, *A Course in Exterior Ballistics* (War Department Document No. 1051, December, 1920) by R. S. Hoar, and in *Computation of Firing Tables for the U. S. Army* (Aberdeen Proving Ground) by H. P. Hitchcock.

The U. S. Army now uses two sets of wind weighting factors, one for ordinary terrestrial fire (this being the same as the one given here) and one for antiaircraft and other high-angle fire. These weighting factors are given in Tables I and III, *Technical Regulations No 1236-1* (U. S. War Department, June, 1934).

## Wind Weighting Factors

Maximum ordinate (feet)	Zones (feet)										
	0-600	600-1,500	1,500-3,000	3,000-4,500	4,500-6,000	6,000-9,000	9,000-12,000	12,000-15,000	15,000-18,000	18,000-24,000	24,000-30,000
600	1.00	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
1,500	.33	.67	.....	.....	.....	.....	.....	.....	.....	.....	.....
3,000	.17	.24	.59	.....	.....	.....	.....	.....	.....	.....	.....
4,500	.11	.17	.26	.46	.....	.....	.....	.....	.....	.....	.....
6,000	.09	.12	.20	.20	.39	.....	.....	.....	.....	.....	.....
9,000	.06	.08	.14	.13	.13	.46	.....	.....	.....	.....	.....
12,000	.04	.07	.10	.10	.10	.20	.39	.....	.....	.....	.....
15,000	.04	.05	.08	.08	.08	.16	.16	.35	.....	.....	.....
18,000	.03	.04	.07	.07	.07	.13	.13	.14	.32	.....	.....
24,000	.02	.04	.05	.05	.05	.10	.10	.10	.10	.39	.....
30,000	.02	.03	.04	.04	.04	.08	.08	.08	.08	.16	.35

**Determination of ballistic wind** 1115. The following example illustrates a convenient method of determining the ballistic wind from aloft wind observations, by means of the above table.

*Given:* Aloft winds have been measured as follows (directions are the true compass directions *from* which the wind is blowing).

<u>Zone (feet)</u>	<u>Velocity (knots)</u>	<u>Direction</u>
0-600	15	270°
600-1,500	12	250°
1,500-3,000	15	260°
3,000-4,500	17	270°
4,500-6,000	22	290°
6,000-9,000	26	320°

*Find:* (a) The ballistic wind for a trajectory whose maximum ordinate is 9,000 feet. (b) The range wind and cross wind corresponding to this ballistic wind and to a line of fire whose true compass direction is 100°.

We shall first weight the winds for the several zones as follows:

<u>Zone</u>	<u>Observed velocity</u>	<u>Weighting factor</u>	<u>Weighted velocity</u>
0-600	15	.06	0.9
600-1,500	12	.08	1.0
1,500-3,000	15	.14	2.1
3,000-4,500	17	.13	2.2
4,500-6,000	22	.13	2.9
6,000-9,000	26	.46	12.0

The weighted velocities for the several zones may now be plotted as vectors, using the observed directions of the winds for their respective zones. A mooring-board diagram, or any other form of polar plotting sheet, is convenient for this purpose. Figure 25 illustrates the method of plotting the vectors. The closing vector, *OW*, of the polygon represents the ballistic wind both as to velocity and as to direction. (Since it is customary in stating wind directions to give the direction *from* which the wind is blowing, care must be taken to plot the wind vectors correctly, i.e., with their arrow-heads pointing in the direction *toward* which the wind is blowing.)

The ballistic wind, as found above, is practically 19 knots *from* 300° true. The direction of the line of fire being 100° true, we find that the wind vector

makes an angle of  $20^\circ$  with the line of fire. The component in the line of fire (range wind) is therefore  $19 \cos 20^\circ = 17.9$  knots, and the component perpendicular to the line of fire (cross wind) is  $19 \sin 20^\circ = 6.5$  knots; both of the components are positive. The same result can be obtained graphically, by dropping a perpendicular from the extremity of the ballistic-wind vector  $OW$  to the line of fire, as shown in the figure;  $OF$  is the range wind, and  $FW$  the cross wind.

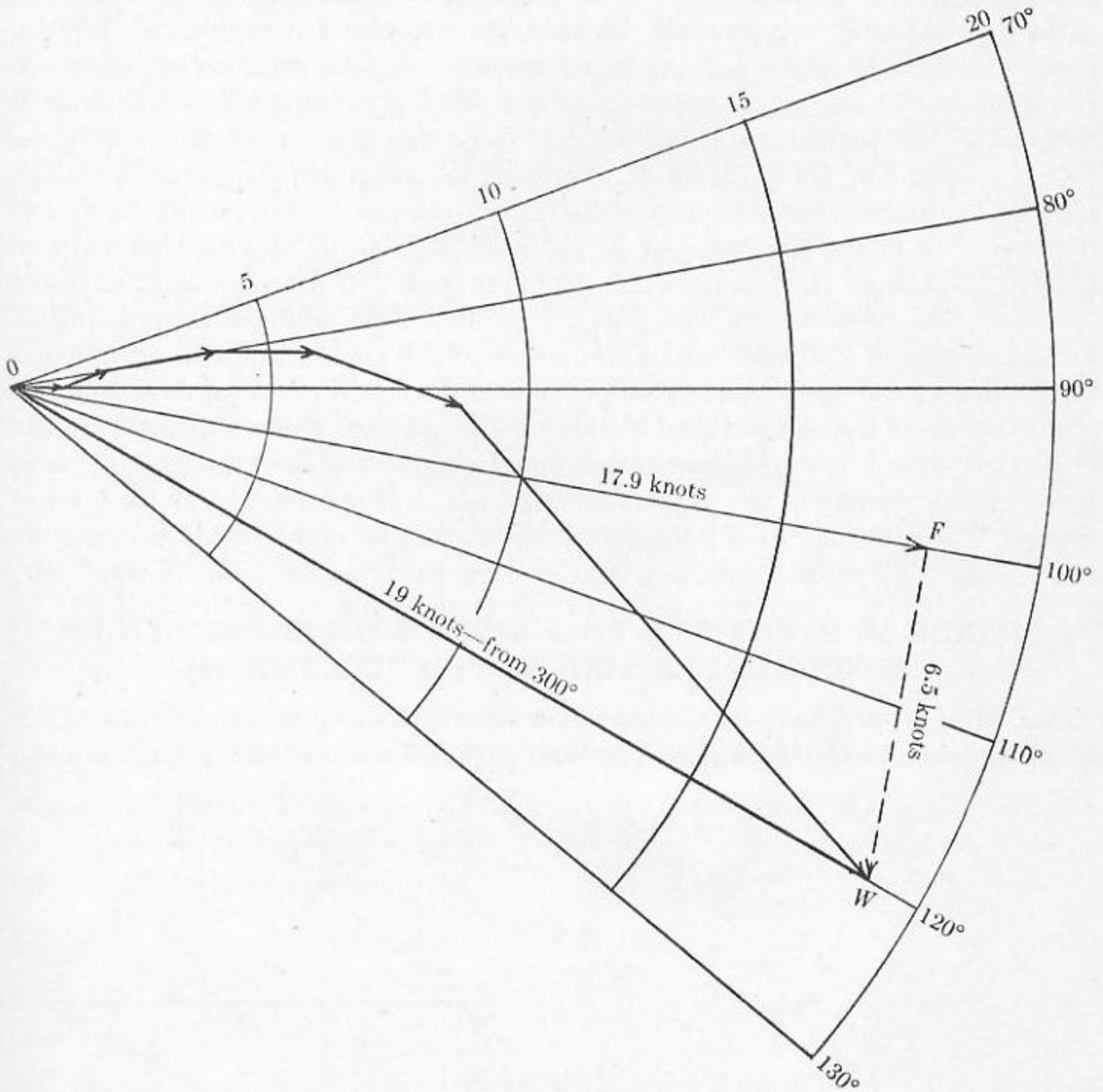


FIGURE 25

1116. It is evidently also possible to resolve the wind for each zone into its range component and cross component, and to weight these components and thus obtain a ballistic range component and ballistic cross component. This method, however, has the disadvantage of delaying the solution for the ballistic wind until the direction of the line of fire is known, and in naval gunnery this is an important disadvantage.\* By means of the method illustrated above, the ballistic wind can be determined immediately after aloft soundings have been taken, and all that remains to be done when the direction of fire becomes known is to resolve this single wind into its two components. Special facilities for accomplishing this

\* It is doubtful whether any real advantage is to be gained, in any case, by the elaboration of using different weighting factors for the determination of ballistic range wind and ballistic cross wind.

rapidly are provided in plotting rooms or control stations; in the latest equipment, the wind can be applied directly to the range keeper, which automatically resolves it into the required components and generates the corresponding corrections according to Columns 13 and 16 of the range table.

The principal point to be grasped at this time is that the ballistic wind is used with Columns 13 and 16 just as any other wind would be used, and that it ordinarily gives a much more nearly correct result than can be obtained with the surface wind alone. For example, let us suppose that a battery of 16" 2600 f.s. guns is to be fired under the conditions assumed for the problem solved in the foregoing article; the maximum ordinate of 9,000 feet corresponds to a horizontal range of 27,200 yards for this battery. At this range the wind component in the line of fire (17.9 knots) increases the range by 213 yards (Col. 13), and the component perpendicular to the line of fire (6.5 knots) causes a lateral deviation of 50 yards to the *right* (Col. 16). Had we used simply the surface wind (15 knots from  $270^\circ$ ), we would have found a component of 14.8 knots in the line of fire and a corresponding increase in range of 176 yards, and a component of 2.6 knots (minus) perpendicular to the line of fire and a corresponding lateral deviation of 20 yards to the *left*. Thus the use of the surface wind in this case would have made matters worse, in deflection, than if the wind had been ignored altogether. The example cited here by no means represents an exaggerated situation. It is not uncommon for the surface wind to differ materially from aloft winds, and for corrections based only on the surface wind to be opposite in sign to those based on the ballistic wind.

Errors  
resulting  
from use  
of surface  
wind alone

#### CHANGE IN RANGE DUE TO A GIVEN COMPONENT OF GUN MOTION IN THE LINE OF FIRE (COLUMN 14).

1117. Let us denote by  $G_x$  the component of gun motion in the line of fire (positive when in the direction of fire and negative when contrary thereto), and

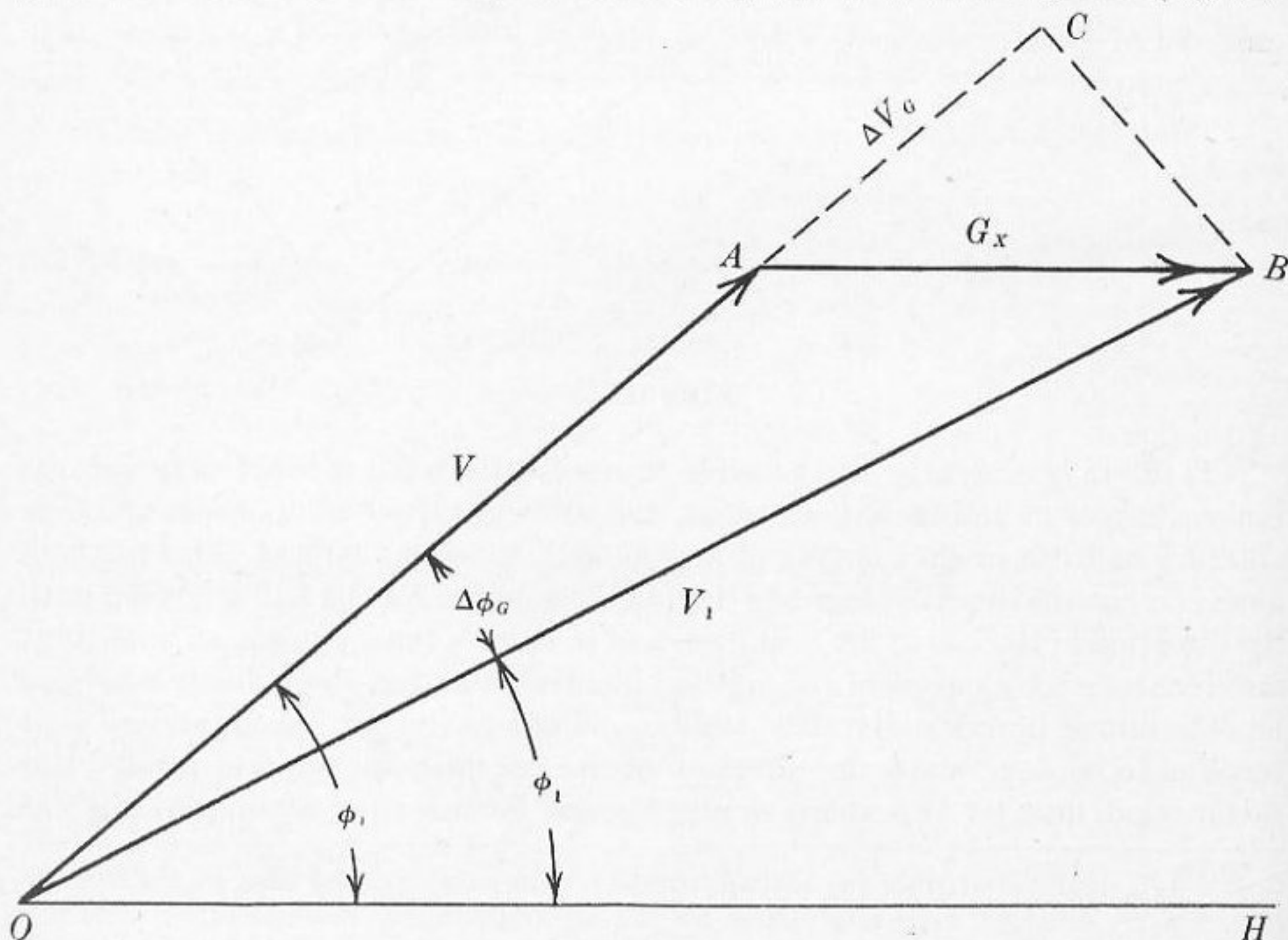


FIGURE 26

by  $\Delta X_G$  the change in range resulting from such motion. A vector analysis of the effect of gun motion on the initial motion imparted to the projectile shows that a positive component of gun motion causes an increase in initial velocity but also an apparent decrease in angle of departure, while a negative component causes a decrease in initial velocity which is accompanied by an apparent increase in angle of departure. The situation for a positive component of gun motion is illustrated in Figure 26. The construction of this figure is similar to that of 23 (b), and by the same process of reasoning that has already been applied in connection with the latter (arts. 1108-1109), we find that the changes in initial velocity and angle of departure in the present case are, approximately,

$$\Delta V_G = G_X \cos \phi \tag{1114}$$

$$\sin \Delta\phi_G = \frac{G_X \sin \phi}{V} \tag{1115}$$

and that in the case of positive gun motion  $\Delta V_G$  is positive and  $\Delta\phi_G$  negative, while in the case of negative gun motion  $\Delta V_G$  is negative and  $\Delta\phi_G$  positive.

If we let  $\Delta X_V$  denote the change in range due to  $\Delta V_G$ , and  $\Delta X_\phi$  the change due to  $\Delta\phi_G$ , we may write

$$\Delta X_G = \Delta X_V - \Delta X_\phi \tag{1116}$$

in which  $\Delta X_G$  takes the same sign as the component of gun motion itself. Comparing (1109) and (1116), we can establish a relation between  $\Delta X_G$  and  $\Delta X_W$  as follows. Substituting (1116) in (1109) we have

Formula for determination of effect of gun motion on range

$$\Delta X_W = (\text{Col. 15}) - \Delta X_G$$

whence

$$\Delta X_G = (\text{Col. 15}) - \Delta X_W.$$

Since all of the range-table columns concerned are based on components of equal magnitude (10 knots), and since  $\Delta X_W$  for a 10-knot component is given by Column 13, we may write finally

$$\Delta X_G \text{ (or Col. 14)} = (\text{Col. 15}) - (\text{Col. 13}). \tag{1117}$$

1118. Mathematical analysis therefore leads to the conclusion, which follows from a literal interpretation of formula (1117), that the effect of a given component of gun motion is the equivalent of the effect of an equal component of target motion less the effect of an equal component of wind. This conclusion is further substantiated by reasoning, as follows. The projectile, entirely apart from the velocity imparted to it by the charge in the gun, acquires from the moving gun a horizontal motion  $G_X$  which, if unopposed, would result in a change of range equal to  $G_X \times T$  in a time of flight  $T$ ; and for equal components,  $G_X \times T = T_X \times T = \text{Col. 15}$ . But the motion thus acquired by the projectile is, in fact, opposed by air resistance, and the resultant change in range is accordingly less than  $G_X \times T$  (or Col. 15). The effect of the air resistance upon the motion  $G_X$  initially acquired by the projectile may be regarded as the equivalent of the effect of an *apparent wind*,  $W_X$ , equal in magnitude but opposite in direction to  $G_X$ —that is, the equivalent of the value given in Column 13 of the range table.

The conception that the motion of projectile acquired from the moving gun is opposed by an apparent wind is more readily grasped in the case of gun motion across the line of fire, since in this case the projectile's motion as acquired from

Analysis of relation among effects of gun motion, wind, and target motion

the motion of the gun can more readily be separated from its motion as acquired from the charge in the gun. Let us suppose, for example, that a gun is fired abeam to starboard from a ship steaming at 10 knots, and that there is no actual wind. Now although it has been specified that there is no *actual* wind, the projectile nevertheless will encounter an *apparent* wind of 10 knots from its left, just as a person standing on the ship and facing to starboard will feel an apparent wind of 10 knots from his left. Since the projectile, upon leaving the gun, has acquired a sidewise motion due to the motion of the gun across the line of fire, it will continue to experience an apparent wind equal in magnitude to and opposing this sidewise motion.

The relation among the changes in range due to gun motion (Col. 14), target motion (Col. 15), and wind (Col. 13), as expressed by formula (1117), should be borne in mind, for it has an important bearing on the practical use of these columns of the range table, as will be brought out in Chapter 12. The same applies to a similar relation among the lateral deviations due to gun motion (Col. 17), target motion (Col. 18), and wind (Col. 16), which will be dealt with shortly.

1119. The computation of values for Column 14 is illustrated in the following example.

*Given:* In the computation of range-table values for the 16" 2600 f.s. gun, for  $\phi = 25^\circ$ , the values for Columns 13 and 15 have been found to be, respectively, 157 yards and 325 yards.

*Find:* The change in range due to a 10-knot component of gun motion in the line of fire (Col. 14).

Applying formula (1117), we have

$$\Delta X_G = 325 - 157 = \underline{168 \text{ yards.}^*}$$

#### LATERAL DEVIATION DUE TO A GIVEN COMPONENT OF GUN MOTION PERPENDICULAR TO THE LINE OF FIRE (COLUMN 17).

1120. We shall denote by  $G_z$  the component of gun motion perpendicular to the line of fire (positive when to the right and negative when to the left with respect to the direction of fire), and by  $D_G$  the lateral deviation resulting from such motion. Figure 24 (b) can be used to represent the vector analysis for the case of gun motion, by substituting  $G_z$  for  $W_z$ . The resultant of the vectors  $OA$  and  $G_z$  is then  $OB$ , as before, and the essential difference between  $OA$  and  $OB$  is the lateral angular displacement of  $OB$  with respect to  $OA$ , which is measured in the horizontal plane by the angle  $DOE = \gamma$ . From the construction we have

$$\tan \gamma = \frac{G_z}{V \cos \phi}$$

Formula for  
determination  
of lateral  
deviation due  
to gun motion

and in a range  $X$ , this angular displacement will result in the lateral deviation

\* The fact that Column 14, in the range tables, is not always exactly equal to the difference between Columns 15 and 13, is due to the fact that it has been customary to compute Column 14 from formulas (1114), (1115), and (1116). This should not of itself change the relation among the three columns, as stated above and as used here for computing Column 14; it does, however, occasion slight computational differences, and differences which arise from the smoothing operations which are applied separately to the three columns. On the whole, nothing at all is to be gained by the greater labor of computing Column 14 from the formulas. The same applies to the slight discrepancies that are found between Column 17 and the difference between Columns 18 and 16.

$$D_G = \frac{XG_z}{V \cos \phi} \quad (1118)$$

Comparing (1118) with (1112), and following through the steps already indicated in article 1117, we find

$$D_W = (\text{Col. 18}) - D_G$$

whence

$$D_G = (\text{Col. 18}) - D_W$$

and finally

$$D_G \text{ (or Col. 17)} = (\text{Col. 18}) - (\text{Col. 16}). \quad (1119)$$

Mathematical analysis therefore leads to the same conclusion in the case of gun motion *across* the line of fire as in the case of gun motion *in* the line of fire; logical support for this conclusion as applied to the present case has already been given in article 1118.

1121. The computation of values for Column 17 is illustrated in the following example.

Computation  
of Column 17

*Given:* In the computation of range-table values for the 16"2600 f.s. gun, for  $\phi = 25^\circ$ , the values for Columns 16 and 18 have been found to be, respectively, 101 yards and 325 yards.

*Find:* The lateral deviation due to a 10-knot component of gun motion perpendicular to the line of fire (Col. 17).

Applying formula (1119), we have

$$D_G = 325 - 101 = \underline{224 \text{ yards.}^*}$$

### EXERCISES

1. *Given:* The range, and the angle of departure and values from Columns 2 (b), 10, and 15 of the range table corresponding thereto.

*Find:* The change in range due to a 10-knot component of wind in the line of fire (Col. 13).

		Given					Answers
		X (yds.)	$\phi$	Col. 2 (b)	Col. 10 (yds.)	Col. 15 (yds.)	$\Delta X_w$ (Col. 13) (yards)
16" 2600f.s.	A	24000	16° 07'	6.4	140	223	93
	B	30000	23 19	8.2	173	307	146
	C	32000	26 11	9.1	185	338	166
	D	38000	38 04	17.4	230	463	236
	E	39000	41 20	25.9	240	493	245

2. *Given:* The range, and the angle of departure and time of flight corresponding thereto.

*Find:* The lateral deviation due to a 10-knot component of wind perpendicular to the line of fire (Col. 16).

\* See note adjacent to art. 1119.

	Given			Answers	
		$X$ (yards)	$\phi$	$D_W$ (Col. 16) (yards)	
A	5''3150 f.s.	5000	2° 00' 9	6.23	8
B		8000	4 14.0	12.02	25
C		10000	6 26.2	17.01	42
D	16''2600 f.s.	3000	1 17.9	3.61	1
E		5000	2 13.9	6.19	2
F		8000	3 47.9	10.31	6
G		12000	6 11.2	16.42	14
H		20000	12 14.8	30.96	41

3. *Given:* The range, and the time of flight corresponding thereto.

*Find:* The change in range and lateral deviation due, respectively, to 10-knot components of target motion in the line of fire (Col. 15), and perpendicular to the line of fire (Col. 18).

	Given		Answers		
		$X$ (yards)	$T$ (sec.)	$\Delta X_T$ (Col. 15) (yards)	$D_T$ (Col. 18) (yards)
A	5''3150 f.s.	5000	6.23	35	35
B		8000	12.02	68	68
C		10000	17.01	96	96
D	16''2600 f.s.	3000	3.61	20	20
E		5000	6.19	35	35
F		8000	10.31	58	58
G		12000	16.42	92	92
H		20000	30.96	174	174